Timed Modal Epistemic Logic

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Chapter 1

Introduction

In these decades, the modal approach to epistemic logic, MEL, has become an indispensable technical tool of research in Computer Science. It has proved helpful for formal reasoning about the distributed systems in particular (e.g. Parikh and Ramanujam (1985); Fischer and Immerman (1986); Halpern and Moses (1990)) and for the modeling of the intentional level of multiagent systems in general (e.g. Fagin et al. (1995a); Wooldridge (2009)). But it is also well-known that it suffers from the notorious logical omniscience problem (see, e.g., Parikh (1987, 1995)). It has been noticed since the beginning of the introduction that agents modeled by the modal approach of epistemic logic are ideal reasoners, who possess unrealistic reasoning ability that a normal agent, human being or machine designed to match human intelligence, is impossible to own. As a result, the applicability of MEL is strictly limited. Many cases in the ordinary scenarios that are supposed to be analyzed by epistemic logical formalisms can't be adequately formulated in the modal approach of epistemic logic.

One way to escape from this undesirable consequence, and therefore enhance the applicability, is to design a logical framework in which the resources consumed in the course of reasoning can be explicitly expressed. Recently a well-studied logical formalism, Justification Logic (Artemov (2006, 2008); Fitting (2005a,b)), which is developed from the study of Logic of Proofs, LP (Artemov (1995, 2001)), is a prominent example. In Justification Logic a formula and the justification that supports the formula are both explicitly expressed. With the information of the justificatory complexity of the formula, we can decide if the formula is possible to be known by the modeled agent in normal circumstances. More importantly, Justification Logic bears a formal connection with MEL. This connection has Justification Logic not only important as a logic of an epistemic concept in its own right, but also a formalism that reveals the underlying logical structure of MEL, which from the beginning of the study of epistemic logic, thanks to the contribution of philosophers, has been the standard vehicle for the reasoning about epistemic attitudes. One of the main theorems concerning Justification Logic is the *realization theorem*, which states that modal epistemic logical theorems are exactly those that can be converted to theorems in Justification Logic with suitable justification terms substituting for modal knowledge operators. Interpreted epistemically, the theorem shows that there is indeed justification structure embedded in modal epistemic logic, which conforms to our informal understanding of knowledge involving justification, and this structure is only explicitly disclosed in the formalism of Justification Logic.

The complexity of justification is not the only data sensitive to the process of reasoning. The temporal span is another and probably a more fundamental resource to be consumed when reasoning takes place. Reasoning takes time to happen, and with the measurement of the time spent in the course of reasoning we can distinguish what is difficult to know from what is easier, and hence resolve the problem of logical omniscience. One goal of this thesis is to bring out a novel logical framework, called timed Modal Epistemic Logic, tMEL, which is originated from the study of Justification Logic, and designed to reason about knowledge and the time of reasoning. Numerical labels are employed to denote when a propositional statement is known by the modeled agent, and for each possible world of a model, it is equipped with a propositional evaluation to indicate the basic facts of the world, and a syntactical device to represent the agent's reasoning process in the world. Axiomatic proof systems will be provided and accompanied with the completeness result. Details are discussed in the content.

In Artemov and Kuznets (2006, 2009), a computational complexity treatment of the logical omniscience problem is proposed, in which an epistemic logical system is said to be not logically omniscient if, roughly speaking, a valid knowledge assertion of a formula contains proof-relevant information of the formula such that a proof of the formula in the system is polynomially bounded by the size of the knowledge assertion. It has been shown that modal epistemic logic can't pass such a test, while Justification Logic, with justification terms as footprints of the real proofs, can. For tMEL, since in many real world applications, the temporal considerations are the main concerns, it certainly has its applicable domains. Moreover, if we use unary representation of the time labels, which is employed to denote the natural ticks of clock but also indicates the hight of real proofs, tMEL should also pass the test with a lighter machinery.

As we've mentioned, the realization theorem holds between MEL and Justification Logic. A similar result is also true for the relation between MEL and tMEL. Theorems in modal logic are exactly tMEL theorems without the numerical labels to indicate when a known proposition is known. Speaking epistemically, this shows that there is indeed underlying temporal structure of MEL. Syntactically, these realization results show the correlations between theorems in these logical frameworks, and given the axiomatic proof systems of these logical frameworks, we can actually extend these realization results in some way to relations between proofs in these systems. This investigation into the realization relations between proofs in these systems is the other topic of this thesis. We will first determine a proper subclass of axiomatic proofs in modal logic, called non-circular, and prove that the class of proofs is complete in the sense that every modal logical theorem has a non-circular proof. This result gives us a structural property of axiomatic proofs, also called Hilbert style proofs, a subject rarely studied in the proof-theoretical literature. We then show that by establishing suitable numerical labels for each modal occurrence, these non-circular proofs are precisely those proofs that can be converted to tMEL proofs. Finally, we prove that there is a two-way translation between tMEL and Justification Logic. Combining these results together, we will establish a formal connection between non-circular proofs and proofs in Justification Logic, and the whole procedure of these proof realization results will also give us an alternative algorithm for the realization between theorems in modal logic and Justification Logic.

In this thesis, the discussions will mainly surround the analysis of the modal logic S4, one of the most discussed and applied epistemic logical systems. That is, our discussion will mostly be devoted to the semantic construction of the tMEL counterpart of S4, called tS4, and elaborate in-depth the formal proof relations between S4, tS4, or called $S4^{\Delta}$ when the axiom system is concerned, and LP. However, the methods provided here are not limited to the analysis of S4 and its tMEL and Justification Logic counterparts. A generalization of these methods will also be addressed.

The structure of this thesis is built up from three parts. The first part, which comprises the next two chapters, is a survey of epistemic logic. Epistemic logic was first introduced by philosophers, and later found its empirical applications in fields such as Computer Science and Economics. Our survey will trace its development from its beginning to its recent advancement. That is, both the philosophical debates, in Chapter 2, and applications, in Chapter 3, will be discussed. The second part, Chapter 4, is for the introduction of tMEL, timed Modal Epistemic Logic. We will also argue in this part how a logical framework reasoning about both knowledge and the time of reasoning can help to resolve the problem of logical omniscience. Finally, in Chapter 5, the last part, we will discuss the logical property of modal logical proofs and then show the formal syntactical relations between proofs in MEL, tMEL and Justification Logic.

Chapter 2

Epistemic Logic: The Philosophical Development

If we look back on the development of epistemic logic, it can be easily recognized that there are two different phases of the development where the landscapes of research are various. For the first, the study is mostly philosophical, with methodologies such as conceptual analysis and others that philosophers are familiar with. This phase, and hence the whole development of epistemic logic, began with the notice of systematical features in our daily usage of epistemic terms, such as knowledge and belief; and then syntactical and semantical logical systems of these terms were introduced, following by philosophical debates over the adequacy of the proposed principles. The ultimate goal here, as a branch of philosophy, is, likewise, to explore concepts of philosophical interest. It is expected that for each epistemic notion a unique correct epistemic logical system can be established and the inherited logical structures of epistemic notions can be faithfully characterized.

Starting in the early fifties of the last century, the philosophical study of epistemic logic developed in a steady path in the next two decades; but we could see the decline of the development at the end of the seventies. While philosophers lost their interest, researchers in other fields, notably Artificial Intelligence, Computer Science and Economics, began to recognize the significance of investigation into epistemic concepts. They started to search for formal models for knowledge and belief, and naturally turned to the philosophical innovations. Research in these fields is application-oriented; hence the existence of correct epistemic logics is no longer the main concern. Rather, logics equipped with different logical axioms, standing for different epistemic principles, can survive simultaneously, as long as these logics have their own applicable domains. In this ongoing developing phase, novel formal mechanisms are introduced for handling various questions raised from practical concerns, and technical results such as completeness, decidability and complexity of these systems are also pursued in the preparation for the possible implementations and for the better understanding of these proposed epistemic systems.

As we can see, the goals and methodologies in these two phases of epistemic logical study are rather different; but there is one question troubling researchers persistently since the inauguration of epistemic logic. The problem of logical omniscience points out that some unrealistic aspects exist in the standard formalism of epistemic logic, and no normal rational beings or machines designed to match human intelligence are able to hold such reasoning ability as construed in these logics. The resilience of the problem has made some philosophers question even the very idea of epistemic logic itself and, for the purpose of application, the search for formal alternative models free from the problem of logical omniscience has never ended.

In this part of the thesis, including this and the next chapter, we will give a survey of the development of epistemic logic and put some emphasis on the study of the problem of logical omniscience. The main goal is to demonstrate important features of the study of epistemic logic such that the discussions and analysis of the logical omniscience problem can be appreciated and evaluated. We will survey from the inception of epistemic logic to its current state; that is, both the early philosophical study as well as the later applications will be covered[1]. It is also hoped that such a survey can benefit both philosophers and application minded researchers. Although the philosophical examinations might not have direct impact on the recent technical and applicational developments in the logical studies of epistemic notions, it is always illuminating to see how the commonly accepted principles are intelligibly questioned; and glamorous philosophical ideas are always the sources of novel research subjects. Furthermore, investigation into the range and limitation of the applicability of epistemic models raised from practical concerns should also deepen our understanding of epistemic notions. Recently, there are efforts trying to revive the philosophical study of epistemic logic and reunite epistemic logic and epistemology, which have separated for quite some time.[2] This work, by looking back the development, should add a contribution to the trend. Now we start our discussions from the very begining.

[1]Halpern (1995), Battigalli and Bonanno (1999), Van Der Hoek et al. (2002), Meyer (2004), and Gochet and Gribomont (2004) are some of the examples of contemporary surveys, most of which focus on the recent technical and applicational development of epistemic logic.

[2]See van Benthem (2006); Hendricks and Symons (2006).

2.1 Von Wright and Hintikka

Although the theme of this thesis is on the contemporary development of epistemic logic, it is worth noting that some relevant work has been done in the Middle Ages.[3] The distinctive feature of epistemic logic as it is today is surely its adoption of the modern logical techniques and notations developed at the turn of the last century. The first deductive logical system with an intended epistemic interpretation was introduced by the Swedish-Finnish philosopher G. H. von Wright in his 1951's seminal work *An Essay in Modal Logic*.

As the title of the book suggests, from the beginning the inquiry into epistemic logic was approached from the perspective of modal logic. Originally devoted to necessity, possibility and other similar concepts, the study of modal logic, in its modern logical form, was started in the 1930's. Several modal deductive systems were then proposed, presented in a style, nevertheless, not much practiced afterward.[4] In contrast, modal logic is usually conducted as an extended propositional logic, with the formal language built up from an additional modal operator (or sometimes two dual operators), and with the deductive systems presented in the way that extra axioms and rules in which the modal operator essentially takes part are added to a complete propositional logical system. Thus on top of propositional logic, von Wright proposed an epistemic system included the following modal axioms:

$K(p \to q) \to (Kp \to Kq)$ *K Axiom, Distribution Axiom, Deductive Closure Principle*

$Kp \to p$ *T Axiom, Truth Axiom, Factivity Axiom*

and the modal rule:

if p is provable ($\vdash p$) then so is Kp ($\vdash Kp$) *Logical Truth Closure*

Von Wright called this system M, and we now call it system T, where K is the modality.[5] The phrases after the axioms and rule are some of their common names.

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[3] Boh (1993) and Knuuttila (1993).

[4] In Lewis and Langford (1932), modal systems from S1 to S5 were introduced as logics of *strict implication*, not of *necessity* simpliciter. Strict implication, as explicated by Lewis, is the *necessity of material implication*. Gödel later reformulated Lewis's S4 in the style we see below. The modality B is employed as standing for *beweisbar*, namely *provable*. That is, Gödel construed S4 as a logic of provability. He then provided an embedding into S4 from the intuitionistic logic; this in turn gave a provability reading to the latter Gödel (1933). For more of the early history of the development of modal logic, readers can refer to Goldblatt (2003), and Hughes and Cresswell (1968).

[5] We have changed the formation of these axioms from von Wright's original texts to have them more easily accessible. Also see footnote 7 below.

This style of presentation has clear strengths. Deductive systems are introduced in a modular way (by adding modal axioms and rules), and it is convenient to provide interpretations to systems (by reading the modal operator differently), or, the other way around, to bring out systems with intended interpretations. This is also how around the same time deontic logic, the logic of obligation, and tense logic, the logic of temporary concepts, were introduced.[6]

Here our modality K stands for knowledge, and Kp means that p is the case is known. That is, it is the propositional knowledge, "knowing that," to be investigated here. The modal axioms and modal rule are the claimed epistemic principles. Readers can find their meanings by reading for themselves.[7] These principles are all quite natural and intuitive, at least at first sight; nonetheless, none of them can be exempted from further examination, as we will see later.

Von Wright also considered the epistemic reading of the following commonly discussed modal axioms.

$Kp \rightarrow KKp$ *4 Axiom, KK-Thesis, Positive Introspection Axiom*
$\sim Kp \rightarrow K \sim Kp$ *5 Axiom, Wisdom Axiom, Negative Introspection Axiom*

He thought the first, while reading epistemically, knowing p implying knowing that of knowing p, is dubious, but the second, not knowing p implying knowing the fact of not knowing p, is absolutely unacceptable.[8] The system combining the system of T with the axiom of 4 is the so-called *S4 system*, and adding 5 Axiom into S4, we will get the *system of S5*. Despite von Wright's negative comments on the above two axioms, the systems S4 and S5, together with the system of T, are the most studied logics of knowledge. Von Wright also noted that in his discussion, the concept of knowledge is in a sense *absolute*, that is, knowledge is not relative to an individual.[9] He suggested that a system with modalities such as "known to a" can be developed, and this is exactly what is in the later development of epistemic logic, where modalities K_a are introduced with $K_a p$ meaning that the individual a knows that p is the case. Thus in the following, only the relative concepts of epis-

[6] Von Wright is also one of the founders of deontic logic (von Wright, 1951b,a), and Prior introduced tense logic (Prior, 1957).

[7] In von Wright (1951a), it is, however, the modal operator *verification* V (together with *falsification* F, equivalent to $\sim V$) to be introduced. It is interesting to see that if an epistemic modality is interpreted in this way, it is not too far from Gödel's construal of S4 as logic of provability.(cf. footnote 4). In the development of Justification Logic (Logic of Proof) (Artemov, 2001, 2008; Fitting, 2005a), a similar parallelism is made between justifications and proofs.

[8] See von Wright (1951a, p. 71 and p. 77).

[9] von Wright (1951a, p. 35).

temic notions will be considered, although if the context is clear, especially when only one knower is concerned, the subscript of the epistemic modality will be omitted.

About ten years later, another Finn, J. Hintikka published the monograph *Knowledge and Belief: An Introduction to the Logic of the Two Notions*, which is arguably the most influential work in epistemic logic. The primary contribution of the work is its pioneering semantic analysis of epistemic concepts. The idea of the semantics, nonetheless, is simple and direct. Given an epistemic attitude, say, knowledge, and a holder of the attitude in question, a talk of the attitude of the holder is to postulate a set, or space, of situations or scenarios or states of affairs or courses of events or, somewhat misleadingly in an epistemic context but popular in use, possible worlds, and, relative to each world w, divide the set into two parts: one consists of the worlds the knower in w considers possible, called *alternatives* to w, alternatives because the knower, based on what information he possesses, can't distinguish these worlds and w; and the other otherwise. Thus, to say the knower has the knowledge p in a world is to say that p is the case in all the alternatives to the world. The relation between a world and its alternatives is called *(epistemic) alternativeness relation*.[10]

Readers familiar with modal logic can immediately recognize that this is exactly the *possible world semantics* applied to epistemic concepts,[11] and this application is in some ways more favorable than the one in the original setting tying the semantics with alethic concepts, necessity and possibility. While it is hard to make sense of the alternativeness relations in the alethic context (i.e. the accessibility relations) other than the universal relation (every *possible* world is *possible*), epistemic alternativeness relations manifest the conceivabilities of the holders of epistemic attitudes; and diverse epistemic attitudes are qualified by different conditions set on the relations. In *Knowledge and Belief*, Hintikka suggests that the alternativeness relation for knowledge be *transitive* and *reflexive*, which, according to the established theory of modal logic, is completely categorized by the axioms T and 4, respectively, and hence the propositional part of the semantics Hintikka pro-

[10]In *Knowledge and Belief*, Hintikka didn't construe his semantics in the way we present it here, although later he also employed this ideology (1986). In his exposition, the semantics is built up from model sets, model systems and alternativeness relations, where, to make comparisons, a model set is the set of formulas true in a possible world, a model system is a space of model sets (possible worlds), and the alternativeness relation in a model system is a relation between model sets in the system. Cf. Hughes and Cresswell (1968).

[11]The formal structure of possible world semantics for S4 is given in Section 4.2.1 and for other modal logics is discussed in Section 4.5.

posed for the logic of knowledge corresponds to the system of S4. For belief, Hintikka advises that the alternativeness relation be *transitive* and *serial*, and the result is a semantics for *KD4*, the system derived from substituting for Truth Axiom in S4 the following axiom (B might be a better notation for the modality in this case):

$Kp \rightarrow \sim K \sim p$ *D Axiom, Consistency Axiom.*

As to the possible world semantics for S5, the alternativeness relation is taken to be an *equivalence relation*, i.e., reflexive, transitive, and symmetric. Alternatively, S5 can also be categorized by the semantics with the *universal* relation as the alternativeness relation.[12]

The intuitive appeal of this semantics for epistemic terms is easily recognized, and it was probably a natural step for Hintikka to take to develop this project. It was his mentor von Wright who suggested that epistemic concepts can be systematically studied, and he himself is one of the pioneers developing the relational possible worlds semantics for modal logic.[13] With this background, Hintikka elaborated the epistemic significance of each semantical rule and discussed some philosophical problems which are relevant or might be raised in his treatment of epistemic notions. All these put together constitutes the main body of the seminal work.

This formal semantical analysis of knowledge and belief was, however, a deviation from the view of traditional philosophers, who anticipated that a logical investigation into epistemic concepts should shed some light on our understanding of the concepts, but couldn't justify their anticipations from the artifacts that Hintikka provided. Thus Hintikka's pioneering work was not fully welcomed at the beginning of its publication,[14] and the ensuing philosophical discussions, as Hintikka commented,[15] were mostly on the deductive methods; semantical analysis of epistemic concepts was largely neglected. In one place, Hintikka explained that his work was providing an "explanatory model" with attempts to capture the basic meaning, or the "depth logic" in the analyzed concepts[16]. The purport and significance of formal methods was reassured. Nothing more than applications can justify a formal model.

[12]Hintikka in a later work also discussed the possibility to analyze perception in terms of possible world semantics. See Hintikka (1969).

[13]See Camap (1947), Kanger (1957), Kripke (1959) and Hintikka (1957), and also, for a historical account of the development of possible world semantics, Goldblatt (2003) and Copeland (2002).

[14]See reviews from Castañeda (1964), Lemmon (1965), Chisholm (1963), Deutscher (1966), and White (1965), where Chisholm's is a paper length overview, while Deutscher's and White's provide strong criticisms.

[15]See Hintikka (1986).

[16]Hintikka (1968).

nowadays, semantical methods are prevalent in the study of epistemic logic.

2.2 Philosophical Debates in Epistemic Logic

After von Wright and Hintikka's publications, there were plenty of works in the literature relevant to epistemic logic. But it is not straightforward to delimit the scope of the study, since the distinction between epistemic logic and epistemology is not easy to draw, especially at the time when epistemic logic was first introduced. For example, we can see the tripartite analysis of knowledge formulated in the following logical form:

$$Kp \longleftrightarrow (p \,\&\, Bp \,\&\, Jp)$$

where B and J stand for belief and justification, respectively; or the analysis is broken down further as the following epistemic principles:

$$Kp \rightarrow p$$
$$Kp \rightarrow Bp$$
$$Kp \rightarrow Jp,$$

that is, knowledge is *true*, *believed* and *justified*. Discussions surrounding these principles for the analysis of knowledge are abundant in the literature, and it is especially so after Gettier's famous, or infamous, challenge to the traditional tripartite view of knowledge (1963). One might argue that, since these discussions are about the correctness of these principles, they are in the domain of epistemic logic;[17] but the truth is that most of them are not motivated by epistemic logic, nor aiming at construction of any formal epistemic system. On the other hand, if the establishment of a logical system of the interrelationships between epistemic concepts is the main concern, then these discussions must be important references and the justification of the system. In the following, however, to limit the length of our discussions and be concentrated, we will eye issues raised by epistemic logicians; and if only necessary for comparisons, some relevant epistemological topics will be touched.

One of the genuine problems pertinent to epistemic logic is the *KK-Thesis*, (4 Axiom).[18] After Hintikka explicitly endorsed this thesis by imposing tran-

[17] In Lenzen's survey (1978), discussions on these principles are included.

[18] Synthesis in 1970 held a symposium on KK-Thesis, where Hintikka among others participated (Castañeda, 1970; Ginet, 1970; Hilpinen, 1970; Hintikka, 1970a). The term KK-Thesis was coined by Hilpinen in his contribution to the symposium. He showed, based on the classical analysis of knowledge, if the KK-thesis was correct, then Kp implied $B(Kp)$, which did not necessarily hold. Occasionally, we still can see the discussions on KK-Thesis these days. See Williamson (1999).

sitivity on the epistemic alternativeness relation, many arguments were raised over the principles, and most of them, ranging diversely from commonsensical arguments to highly technical accounts, opposed Hintikka's proposal.[19] However, in his defense, Hintikka pointed out that to many great historical thinkers, when they discussed the importance of knowledge, this thesis was what they had in mind.[20] Here we see a conflict between our different perspectives on the concept of knowledge, and this difference should be what Hintikka had in mind when he insisted that he was discussing a strong sense of knowledge. So knowledge, on the one hand, just like justice, is a virtue to pursue. Its goal is to secure our understanding of the outside world, and hence the standard of having it should be high, so high that the holders of pieces of knowledge should also know that they own the pieces, so as to be able to respond to any challenge raised about their holding of the knowledge. However, on the other hand, if we push this standard hard, then all the knowledge claims having been made so far are not legitimate, a consequence difficult to be plainly accepted.

A relevant philosophical dispute is worth mentioning. After Gettier's criticism the epistemology circle surged to search for a new definition of knowledge, and this led to the debates between *internalism* and *externalism*.[21] These debates are closely related to those about KK-Thesis, in which the concept of knowledge is the main concern, while the internalism/externalism debates bear on justification or relevant concepts, over what condition(s) is (are) necessary to turn true belief into knowledge. Internalism, roughly speaking, argues that a subject has to have *internal* access to whatever justifies his true belief, while externalism contends that this requirement is not essential to converting true belief into knowledge. According to this distinction, it is natural for internalism to be in favor of the KK-Thesis, as the process of internal access is easily equated with the process of knowing, and for externalism to deny that "knowing that one know" is a necessary condition of knowledge; nonetheless, none of these correlations are *a priori* and cannot be otherwise.

[19]The most common argument is that the thesis can't be applied to children or animals, since they don't have the concept of knowledge. Also it is argued that iterating the thesis the knower will have knowledge of any epistemic levels, which is implausible (Cargile, 1970). Others are that in order for the thesis to hold the knower won't forget what he knows (Lemmon, 1967), and needs to have the concept of oneself (Castañeda, 1967), the latter argument carried out in the setting of quantified epistemic logic. Some externalists' counterexamples to internalism can also be taken as against KK-thesis. See the next paragraph.

[20]Hintikka mentioned Plato, Aristotle, Spinoza, Locke, Schopenhauer, and many other big names. See (1962, sec. 5.3), and also (1970a).

[21]See Kornblith (2001) for an up-to-date collection of the debates.

So far our discussions have been only about the propositional epistemic logic, but it is the whole package of quantified epistemic logic which was introduced in *Knowledge and Belief.* The philosophical debates surrounding quantified epistemic logic are the most intensive in epistemic logic.[22] But compared with the expressivity, and the amount of relevant philosophical studies, the applications of quantified epistemic logic are relatively limited. The problem is that before we can have good applications of quantified epistemic logic, important decisions have to be made, and these are usually non-trivial and sometimes confusing.

The major problem that quantified epistemic logic faces is the same as what quantified alethic logic confronts. That is, our commonsensical logical principle, or *Leibniz's Law* as it is usually called,[23] that substituting for each other two identical objects or, in another manner, two names for the same object in a statement results in statements of the same truth value, might fail if we regard statements with objects or names of object within intensional contexts – e.g. "it is necessary that," "know that," or "believe that;" and the fall of the principle also puts in doubt the rule of existential generalization.[24] Counterexamples given, Quine, the most active opponent, attacked severely the modal analysis of these statements.[25] He revived the philosophical query raised by Frege and Russell, but mostly neglected at his time.[26]

The problem of substitutivity in Hintikka's epistemic semantic proposal was echoed by Chisholm, who raised the relevant problem of how to identify individuals across possible worlds.[27] A solution to the problem was suggested by restricting the range of singular term only to *genuine names*, each of which refers to one and the same individual across all over the epistemic

[22]The first volume of Noûs threw a symposium on quantified epistemic logic. Chisholm (1967) and Hintikka (1967b) among others are contributors. In Lenzen (1978), the chapter of quantified epistemic logic is one third of the book.

[23]Leibniz's Law is usually formulated as a biconditional $a{=}b \leftrightarrow (\varphi(a){\leftrightarrow}\varphi(b))$, where $\varphi(a)$ is a statement of a. The relevant issues discussed here are mainly on the problems raised in the direction from left to right, which is sometimes separately named the *principle of indiscernibility of identicals*.

[24]That is the rule from $\varphi(a)$ to $\exists x\varphi(x)$. If the *principle of indiscernibility of identicals* fails for a statement $\varphi(x)$ with intensional context and x is within the scope of the context, then it is possible that $\phi(a)$ is true and $\phi(b)$ false but $a = b$. Then the validity of the rule is questionable.

[25]Counterexamples to the Leibniz's Law, which won't be repeated here, can be found in most works on the relevant topics. In (1948; 1953; 1966), Quine argued that the modal logic for necessity and possibility is impossible, and in (1956), Quine considered the problem of quantifying into epistemic contexts, and provided a syntactical treatment. Also see Kaplan (1968).

[26]*Vide* Frege (1948) and Russell (1905). Also Linsky (1971).

[27]Chisholm (1965, 1967).

alternatives of a world in discussion.[28] By making this amendment, the principle of substitutivity is restored. Similar suggestions were also made for the context bearing on the concepts of necessity and possibility,[29] following by extensive philosophical investigations to expel, e.g., Quine's impeachments, such as the accusation of quantified modal logic presupposing some kind of unwanted essentialism.[30] All these efforts finally led to the direct reference theory of proper name,[31] as opposed to Frege's and Russell's then dominant description theory of reference. Hintikka, on the contrary, didn't accept the idea of restriction on singular terms but insisted that the cross-identification should be feasibly grasped by *individual functions*, functions assigning each world an object in the world. He embraced the failure of substitutivity, and rejected the new theory of reference overall.[32]

In the above a rather sketchy review of quantified epistemic logic is given. The importance of the relevant debates is not completely revealed. These debates are studied in philosophy of mind and epistemology, and central to philosophy of language and even to analytic philosophy in general. Two of the most intricate and aged logical/metaphysical concepts, *identity* and *existence* (in the form of the particular quantifier), are studied in modal discourse. Other important concepts in analytical philosophy such as proper names, rigid designators, natural kinds, definite descriptions, sense and reference are carefully scrutinized. Even restricting focus on epistemic statements, distinctions like de dicto/de re, opaque/transparent, narrow scope/wide scope, notational/ relational, and sensu composito/ sensu diviso are not easily to be clarified, and hence what the object of knowledge and belief is is still under debate. Back to the possible world setting, there is an issue about deciding the domain of quantification, determining whether it ranges over only individuals existing in the world in question, or all individuals, no matter whether actual or possible, of all worlds. This decision will also affect the validity of the famed Barcan and Converse Barcan formula:[33]

$\forall x K\phi \to K\forall x\phi$ Barcan formula
$K\forall x\phi \to \forall x K\phi$ Converse Barcan formula,

where the latter is usually thought less problematic, while the former is considered highly questionable.[34]

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[28]Føllesdal (1967).

[29]See Marcus (1961), Føllesdal (1961) and Kaplan (1978).

[30]Barcan (1947); Føllesdal (1961); Parsons (1969).

[31]See Kripke (1981), and further discussions in Salmon (2005).

[32]Hintikka (1967a,b, 1988); Hintikka and Sandu (1995).

[33]These formulas are first discussed in Barcan (1946), in which Marcus, with her maiden name Barcan, presented the earliest axiomatic quantified modal logical system.

[34]For early objections to Barcan formula see Prior (1957) and Hintikka (1961). Marcus

Hintikka in *Knowledge and Belief* also tackled *Moore's Paradox*, to show that his semantical work can help settle philosophical problems. The paradox is introduced by Moore, who invited us to consider the absurdity in uttering sentences of the kinds: "it is raining but I don't believe it" and "it is raining but I don't know it."[35] Note that these Moore sentences themselves are not the source of the paradox; they can just simply be the case. But what is paradoxical is that the absurdity arises when the utterer of one of these sentences says this of himself in the present tense in an assertive way. Wittgenstein was fascinated with this paradox. He introduced the eponym of the paradox, and once remarked that the discovery of this paradox impressed him most among all Moore's work.[36] Moore's own explanation of the absurdity, roughly speaking, is that an assertion of p *implies* in the ordinary sense the believing of p, while Wittgenstein argued that the first person's belief report that "I believe p is the case" is the same as the assertion that "p is the case." In either proposal, a contradiction follows the uttering of a Moore sentence.[37]

Hintikka, however, came out with a different solution. He argued that when an utterer uttered a sentence, some pragmatic presumptions had to be fulfilled in order to convey information successfully. One of them is that the uttered sentences have to be conceivably believable or knowable, but this presumption is not fulfilled when Moore sentences are uttered. He showed that, as we expect, "$p \mathbin{\&} {\sim}Bp$" and "$p \mathbin{\&} {\sim}Kp$" are consistent, but sentences "$B(p \mathbin{\&} {\sim}Bp)$" and "$K(p \mathbin{\&} {\sim}Kp)$" are not in their respective logics.[38] Though Hintikka's solution is not getting much attention these days,[39] it does turn on a case showing what formal methods can do to help deal with epistemic paradoxes.

(1962) defends the formula. Fitting (1999a) and Fitting and Mendelsohn (1998) analyze the correlation between the semantics setting and the two formulas, where Fitting and Mendelsohn (1998) is also a good modern source for general philosophical and technical issues in quantified modal logic.

[35]The belief version of the paradox is well-known. It is scattered in Moore's writings (Moore, 1942, 1959; Baldwin, 1993), and a variant form that "$p \mathbin{\&} B{\sim}p$" was also considered (1959). On the other hand, the knowledge version of Moore's paradox is less familiar. Moore himself did discuss it, but the relevant document was published in Moore (1962), so Hintikka probably didn't notice that and hence attribute to Moore where the knowledge version of the paradox is concerned in Hintikka (1962).

[36]The phrase *Moore's paradox* first appeared in Wittgenstein (1953), Part II, Section X. Malcolm (1984) noted Wittgenstein's praise.

[37]Also see Wittgenstein's discussions on Moore's paradox in (1980).

[38]Using Hintikka's terminology, he showed that these sentences are *indefensible*. See the next subsection.

[39]See Green and Williams (2007), which is an anthology of recent studies on Moore's paradox.

Beyond its utterance, the peculiarity of the Moore sentence itself is also full of philosophical interests. In the so-called *knowability paradox*, or *Fitch's paradox*, the sentence is taken to show that the verificationist's principle: "if a statement is true, then it must be in principle possible to know that it is true"[40] is logically rejectable. The derivation roughly goes as follows. Since the Moore sentence "p & $\sim Kp$," as Hintikka shows, can't be consistently known, accordingly, based on the verificationist's principle, this sentence can't be the case. And it immediately follows that "$p \rightarrow Kp$", an impossible conclusion that every truth is known. Later we will see that the Moore sentence also plays a crucial role in revealing a stimulating phenomenon of a dynamic version of epistemic logic.

After the publication of *Knowledge and Belief*, *epistemic logic* has been a general term for both logic of knowledge and logic of belief. If a distinction has to be made, the phrase *doxastic logic* will be employed for logic of belief, to match the etymology.[41] In recent literature, we can also find the use of *doxastic logic*, or *logic of belief* as the general term, and this use is not without grounds. Knowledge is usually treated as a special case of belief, namely, true belief with something else, and in application normally the truth condition is less relevant, since what is interested is the reasoners' actions which are caused by their beliefs. Hintikka implicitly accepts the doctrine that *knowledge entails belief*, for the Consistency Axiom, the distinctiveness of the logic of belief that Hintikka proposes,[42] is derivable from the axiom of Truth; however, philosophers have raised doubts about this *entailment thesis*,[43] and an analysis of Moore's paradox adds one more to the list. It has been contended that making an assertion is to *represent* the asserter's knowledge, that a responsible addresser only asserts what he knows.[44] Then based on this account, for a responsible utterer asserting that Moore sentence p & $\sim Bp$, he is meaningfully representing that he knows that p, and at the same time expressing that he does not believe it. Here is a case in which knowing does not entail believing, and the absurdity is yielded since the

[40]This quote is from Dummett (1996). As suggested by Church as an anonymous referee (2009), Fitch (1963) employed Moore's sentence to present a derivation that later Hart and McGinn (1976) regarded as a *reductio ad absurdum* of the verificationist principle. Also see Salerno (2009), Van Benthem (2004), and Dean and Kurokawa (2009).

[41]In Greek origins, *episteme* means knowledge, while *doxasia* means opinion, conviction, or belief.

[42]That our belief system is consistent as this axiom suggests is criticized by Lemmon (1965), Chisholm (1963) and Foley (1979).

[43]The phrase is dubbed by Lehrer in (1968). See Radford (1966, 1967) for arguments against the thesis.

[44]See Unger (1975); Slote (1979); Williamson (1996) for the knowledge account of assertion.

platitudinous thesis is critically challenged.

2.3 The Problem of Logical Omniscience

In the above we give an overview of the early philosophical developments in epistemic logic, relating some renowned debates surrounding the subject. For an early reference to the development in this period, Lenzen's *Recent Work in Epistemic Logic* (1978) is the source. But somewhat paradoxically, the publication of the volume didn't mark the flourishing of the subject; instead, the late seventies was the time when philosophical interests in epistemic logic declined. The reason for the decline has not been formally discussed, but it is not difficult to detect. The major methodologies applied to settle the relevant debates, basically involving practitioners' intuition on examples and counterexamples, seems to lead the arguments to nowhere; and all of the proposed epistemic principles are questioned about their adequacy, which include the prestigious Truth and Distribution Axioms.[45] Epistemic logic was thus regarded as standing on a shaky ground, which kept philosophers away from the subject. Above all is the problem of logical omniscience, which is the most notorious and has been considered as inseparable with any formal construction of epistemic logic.

Generally speaking, the problem of logical omniscience is not a problem of a single axiom or a single rule, and not even only to the modal approach of epistemic logic. It is a problem about the difficulty of having an intuitively appealing epistemic logical system which can capture the knowledge of a rational being, who can reason logically, without leading to an undesirable ideal consequence. For the setting of modal epistemic logic in the tradition of von Wright and Hintikka, the problem can be best manifested by the following rules:

(1) $$\frac{\vdash \phi}{\vdash K\phi}$$ (*Logical Truth Closure*)

(2) $$\frac{\vdash \phi \rightarrow \psi}{\vdash K\phi \rightarrow K\psi}$$ (*Logical Consequence Closure*)

(3) $$\frac{\vdash \phi \leftrightarrow \psi}{\vdash K\phi \leftrightarrow K\psi}$$ (*Logical Equivalence Closure*).

All these rules are derivable in the proposed deductive epistemic systems and

[45] In Armstrong (1953) a pragmatic analysis of knowledge was given, by which the Truth Axiom was redundant. The problem of Distribution Axiom is addressed later in this section.

possible world semantics, but none of them captures the reasoning ability that a normal being is able to reach. Reading them epistemically, these rules say that the knower knows all logical truths, all logical consequences of what he knows, and all logical equivalences of his knowledge.

Before moving on, we should take a look at a relevant but essentially different philosophical problem so as to make a clarification. In the literature the phrase *deductive closure problem* is not well defined. It sometimes refers to the problem around the rule of Logical Consequence Closure above, but mostly, and also in the following discussions, it points to a problem about the Distribution Axiom, or just called *Deductive Closure Principle*, $K(p\rightarrow q)\rightarrow(Kp\rightarrow Kq)$. The main difference is that this problem concerns the situation when the premise implication is *known* to the knower, which doesn't have to be logical truth as required in the rule (2), a condition sometimes just supressed in the texts, and hence confusion arises.

The deductive closure problem was initiated by externalists, who furnished counterexamples to one of the most appealing axioms in epistemic logic, so as to cast doubt upon inferential knowledge, the internalists' prototype of knowledge.[46] Whether the cases raised in discussion were genuine counterexamples or not was debatable,[47] but the Deductive Closure Principle was not thought as obvious as it had been.[48] In particular, the withdrawal of the principle provides a tempting reply to the skepticism.[49] Therefore comes the contextualists who reconciles the diverse intuitions behind the relevant discussions and the merits from views on both sides. For these contextualists, those externalists' counterexamples are granted, and knowledge is still deductively closed, but the context in which the epistemic term is used now has to be taken into account.[50]

The relation between the deductive closure problem and the logical omniscience problem is epitomized by the logical relation between the Deductive Closure Principle and the rule of Logical Consequence Closure. Given that the knower has knowledge of basic logical axioms, it is sure that if knowledge is not closed under one step deduction, or in another way of saying that the

[46] I sketch the famed counterexample given in Dretske (1970) here. If you go to a zoo and see a zebra, you know it is a zebra, but at the same time, considering what kind of evidence you have except that you see a zebra, which should be neutralized, you don't know whether it is a mule "cleverly disguised to look like a zebra." If this is the case, then the *Deductive Closure Principle* fails, since something is a zebra implies it is not a mule, but you don't know that. See more recent discussions in Dretske (2005).

[47] Cf. Vogel (1990).

[48] Now whether knowledge is closed under deduction is one of the *contemporary debates in epistemology*. See the first chapter of Steup and Sosa (2005).

[49] Dretske (1970, 2005), Nozick (1981).

[50] See Cohen (1999), DeRose (1995), and (Steup and Sosa, 2005, chapter 2).

knower fails to apply the rule of Modus Ponens, there is no room for a knower to be logically omniscient, knowing all the logical consequences of his knowledge. But even if we either disregard the externalists' counterexamples, or accept their explanatory force but only apply the Deduction Closure Principle under proper contexts as suggested by contextualists, the problem of logical omniscience still holds. The nature of the problem is about the complexity of logical implications versus the knower's reasoning abilities. Some conclusions of implications are just so removed from their premises that even if a knower has knowledge of the premises, it is unlikely that he has the ability to know their conclusions.

Philosophers had noticed the problem of logical omniscience since the beginning of modern epistemic logic.[51] But compared to its impact, the philosophical examinations of the problem were rather limited. One of the reasons is probably that the employment of formal methods to analyze epistemological problems didn't induce philosophical interest yet, but more likely is that philosophers did not try to address the problem at all, but thought that the problem was unsolvable, and the idea of epistemic logic was a mistake. As such, in the philosophical study of epistemic logic, if a researcher thought there was a responsibility to provide an explanation to the problem, the most common method adopted is the *reinterpretation strategy*, namely, to reinterpret what epistemic logic was about, aside from being about the ordinary epistemic notions that normal human beings can hold.

There are two main ways to explain away the problem of logical omniscience. The first is to regard the enterprise of epistemic logic as being about, instead of knowledge in the ordinary sense, so-called *implicit* knowledge, the knowledge which is not only possessed by the knower currently but also the logical consequences of the knower's knowledge, the knowledge the knower is *in principle* able to know. Thus by the very definition, a logic of this concept of knowledge surely satisfies the above rules (1), (2), (3). This is the simplest and most general strategy used in the literature.[52] But there is a price to pay. Once we understand epistemic logic in this way, it's not clear how this logic can have real applications. In particular, if we hope for the logical study of epistemic concepts which can help us anticipate what a knower will do next, then certainly only the study of the knowledge that the knower currently possesses, not of the implicit knowledge, can fulfill the goal.

The other way is retaining that epistemic logic is a logic of ordinary knowledge, but is best applied to an ideal "rational man," or to a man who

[51]Hintikka certainly noticed this problem. See (Hintikka, 1962, 34-35), but he didn't use the phrase "logical omniscience problem." The phrase is dubbed by Chisholm in his review of Hintikka's (1963).

[52]See Rescher and Nat (1973) and Lenzen (1978).

"follows the consequences of what he knows as far as they lead him."[53] On the surface, this strategy has a better shot. It is about ordinary knowledge, and its real applications depend on how far an ordinary knower like us approximates an ideal man. But the problem is that our reasoning ability is truly distant from ideal knowers, which makes it that only in some simple cases the modal approach of epistemic logic can be successfully applied. In most of cases we are just not as perspicacious as this ideal rational man; we are not about to know immediately, or in a definite finite period of time, all the conclusions of logical derivations provided we know the premises. The applications of epistemic logic under this interpretation are still limited. The logical omniscience problem is not diminished.

Hintikka suggests yet another way of dealing with the logical omniscience problem, which to some extent pertains to the "reinterpretation strategy" as well; but this time it is the meta-logical concepts to be reinterpreted. Concepts such as the *inconsistency*, *consistency*, and *validity* are reinterpreted as *indefensibility*, *defensibility*, and *self-sustenance*, and concepts of *logical implication* and *equivalence* are reinterpreted as *virtual implication* and *equivalence*. Noticing that even given that $p \rightarrow q$ is a logical truth and the knower has knowledge of p, there is no logical reason for the knower to know q, Hintikka contends that, however, if we present the logical argument from p to q to the knower and he is reasonable, then he is not able to defend his not knowing q. Hence Kp & $\sim Kq$ is *indefensible*, and Kp is said to *virtually*, not *logically*, imply Kq. This interpretation is probably the most direct reading of the modal epistemic logic suggested by von Wright and Hintikka (certainly Hintikka himself thinks so).[54] But if this reinterpretation of meta-logical concepts is unavoidable in the constitution of epistemic logic, then it is left to wonder, for some philosophers, in what sense the knowledge operator in epistemic logic is a logical constant; in another words, in what sense epistemic logic is logic at all.[55]

As we can see, these "reinterpretation strategies" do not substantially resolve the logical omniscience problem. The targeted logic these strategies try to interpret is still the one that causes the problem. The best we can get from these strategies is to *make sense* of the epistemic logic, which looks like inseparably living with the problem. To essentially solve the problem, Hintikka did make efforts. He suggested that only when the implication from p to q was a "surface tautology," with some kind of *obviousness*, the rule (2) could be applied. This is certainly the solution that epistemic logicians

[53] The two quotes are from (Lemmon and Henderson, 1959, p.39) and (Hintikka, 1962, p.36), respectively.

[54] See (Hintikka, 1962, Sec. 2.6) and the following sections.

[55] E.g. Hocutt (1972)

dream of, but it is difficult to achieve. The concept of surface tautology seems a subjective term in nature, but in order to have it be applicable, we need a rigorous definition. Hintikka's own proposal, borrowed from von Neumann's idea,[56] is that only when the argument from p to q without going beyond the quantificational depth, the number of nested quantifiers, plus the number of free individuals in p or q, the rule (2) can be applied.[57] How this proposal is relevant to our natural reasoning processes is still arguable. But the main problem is that it only can be applied to the cases in which first order sentences are involved. The logical omniscience problem happens, however, already in the propositional level of epistemic logic. Some propositional derivations are hard enough such that a normal rational being is not able to infer them. Accordingly, if we want to solve the logical omniscience problem following this line of thought, a more general idea of surface tautology or the concept of obviousness has to be found. In Chapter 4, we will see that the logical framework tMEL provides a possibility to define the surface tautology in terms of the time or steps needed to take to derive the tautology.

To conclude this section, the logical omniscience problem in epistemic logic poses a challenge for philosophers. It has been conceded that the concept that epistemic logic is trying to explore needs to be reconsidered. Epistemic logic is then construed as either about a special kind of knowledge of a normal human being, or about ordinary knowledge but of a rational man who has ideal reasoning abilities. But there are also philosophers who are skeptical of the whole project of the logical study of epistemic terms. It is argued that epistemic logic, if there is any, is just not genuinely *epistemic*, if the reinterpretation strategy is inevitable, or not genuinely *logic*, if, as Hintikka suggested, the original meta-logical concepts have to be revised.[58]

[56] Neumann et al. (2000).

[57] See Hintikka (1966, 1970b,c). Hintikka also advised that the impossible world semantics suggested in Rantala (1975) could be developed into a semantics for the logic with restricted quantification depth (Hintikka, 1975). Also Hintikka (1986).

[58] Hocutt (1972).

Chapter 3

Epistemic Logic: Applications

Moving to the late seventies, the dwindling project revived due to the empirical research finding applications in epistemic logic. The study of Artificial Intelligence in particular and Computer Science in general started to recognize the significance of ascribing mental qualities to machines and computer programs.[1] As one of many reasons, the ascription simplifies the expression of valuable information without mentioning the physical details, and such expressions with mental qualities are closer to what the designers have in mind when the devices are constructed. This proposal when it is first raised is not only suggested to apply to machines with intelligence, or designed to be intelligent, which seem more reasonable to be ascribed mental qualities; but also apply to mechanical devices, which can be as radical as a thermostat, *believing* that the room is too hot or not, as long as ascribing mental qualities enhances our understanding of the devices, and then improvements can be made. This treatment of machine and program is close to regard it as what philosopher Dennett called "an intentional system, a system whose behavior is reliably and voluminously predictable via intentional strategy."[2]

[1]See McCarthy (1979, 1983). The idea comes from the observation of our ordinary conversations including such a sentence that, using McCarthy's own automatic teller machine example, "It *thinks* I don't have enough money in my account because it doesn't yet *know* about the deposit I made this morning." Then taking this ascription seriously, discussions about the conditions on when these ascriptions are theoretically adequate and effective are proceeded.

[2]The quote is from Dennett (1987). As a study of philosophy of mind, Dennett distinguishes three strategies that can be employed to predict behavior of entities. One is to treat them as physical entities, predicting behavior by natural law. The second is to treat them as designed objects, predicting their behavior by the functions that they are designed for. The third, and the most abstract level, is to treat them as rational agents, predicting their behavior by their beliefs, desires, their intentionality. This position of philosophy of mind leaves the room for the attribution of mental qualities such as beliefs to objects other than human beings, and from the study of Artificial Intelligence, a similar

Once machines or programs are ascribed with mental qualities, formal models are in need for the study of epistemic properties, and, for the purpose, Hintikka's semantical work of epistemic logic just fits the need, and lays down the foundation for the future work. Two textbooks published in 1995: Fagin, Halpern, Moses and Vardi's *Reasoning About Knowledge*, and Meyer and van der Hoek's *Epistemic Logic for AI and Computer Science*, mark the maturity of the subject. In a discipline where the behavior of rational beings is the subject matter, it seems natural to expect that the logical study of epistemic concepts has its role. And it is true that in Economics and Game Theory in particular, epistemic logic has been put into the toolbox for theorists; it is employed to examine the propriety of epistemic assumptions that are made about the decision-makers or players of a game.[3]

Several distinct features of the applicational study of epistemic logic are worth noting here. Firstly, researchers concerning with applications have a different position on epistemic logic. The interest is no longer in finding out the inner logical structure of epistemic notions, but in revealing the relations between the reasoners' epistemic attitudes and their actions. Researchers are interested in learning what the knowers will come to know, given the reasoning ability that we suppose they hold, and from what they know predict what they will do. The center of research is no more on pursuing the unique correct epistemic logic, but many epistemic logics can coexist as long as they are useful in their own applicable domains.

This pragmatic stance on epistemic logic is reflected in the usage of the term for a knowledge holder from a *knower* to an *agent*, a *doer* in its etymology. It is also reflected in that philosophers' KK-Thesis, a descriptive principle of knowledge, becomes Positive Introspection Axiom, an axiom about a reasoning ability. The axiom of 5, $\sim Kp \to K \sim Kp$, which philosophers reject as an epistemic principle thoroughly, is reintroduced into epistemic logic as Negative Introspection Axiom, another property of reasoning ability. From the applicational point of view this axiom is not adopted without grounds. Things that we don't know and we don't know that we don't know are many in our daily life; however, in a normal applicational scenario, we usually would like to check if agents are able to know some definite propositions in concern, given the reasoning ability that they are attributed to and the informational environment that they belong to. Agents are supposed to be goal-directed, making all their efforts trying to reveal the truth of the propositions; accordingly, if the agents don't have enough information and hence, are unable

idea arises.

[3]See Osborne and Rubinstein (1994); Aumann (1999). More discussions are in the section of Common Knowledge.

to know the propositions after their efforts, they know that they don't have knowledge of it. Given the above reason and the simplicity, intuitive appealing, and mathematical elegance of the associated possible world semantics, in which the alternativeness relation is the equivalence relation, it turns out that the modal logic S5, contrary to its antipathy from philosophers, is an epistemic logic system of applicational researchers' favorite.

There is yet another distinct feature which is much influential. The main concern of epistemic logic is no more on just a single knower but a system of multiple agents. The reason for this change is, however, not hard to conceive. If the aim of epistemic logic is to explore the structure of epistemic notions, then the idea of considering cases of more than one knower is not so inviting; but the most intriguing realistic scenarios in which knowledge or belief plays an essential role are those with more than one agent involved, who can communicate, cooperate or even compete with each other by sharing information. Many of these scenarios are subject to logical investigation. Analysis of such systems of multiple agents will be the main topics of the subsequent discussions. Although models of more than one agent are not formally introduced or discussed before applications of epistemic logic are the focus of research, yet, as long as no new epistemic notion is introduced to intertwine the knowledge of different knowers, the generalization of possible world semantics from a single knower to multiple agents is straightforward.

The problem of logical omniscience is certainly not neglected.[4] Though successful applications has been established, as some of them will be discussed in the following sections, we can still easily come up with realistic examples which are supposed to be analyzed by logical devices of epistemic notions but fail to be so by the modal approach of epistemic logic. Efforts trying to provide epistemic formalisms without the deficiency of the problem never cease. Some of these efforts will be discussed at the end of this chapter.

In the following, most of the technical details such as complexity and expressivity of logical systems, won't be supplied. They can easily be found in many introductory works and textbooks on this topic.[5] Instead, our interest is to explicate the relationship between epistemic logic and its applications. The backgrounds of these applications will be illustrated, and the constructions of the proposed formal settings explicated. The goal is through these discussions that our understanding of these applications can be enhanced, and then further applications of epistemic logic can be developed.

[4] See Parikh (1987, 1995), and other references in Section 3.5.

[5] See, e.g. survey articles Halpern (1995), Van Der Hoek et al. (2002), Meyer (2004), and the textbooks mentioned above.

3.1 Possible World Semantics and Muddy Children

One way to create computing machines or programs with intelligence is to make them hold the capacity of representing and logical reasoning about our external world, as our common sense might tell us this is the distinguished feature of human intelligence.[6] Not surprisingly, the most common formalism adopted by these intelligent agents is first-order language, and hence, for the purpose of extending the representability of intelligent agents to reason about the mental qualities of the other agents, or bringing in a meta-level formalism to reason about the mental behaviors of a group of agents, it would be natural and convenient to introduce a binary predicate $Know$ (or Bel) into the formalism, with $Know(x, \ulcorner\phi\urcorner)$ meaning that the agent x knows ϕ ($Bel(x, \ulcorner\phi\urcorner)$ meaning that the agent x believes ϕ), where $\ulcorner\phi\urcorner$ is the *standard name* for the formula ϕ constructed according to some first order theory which can reason about its own syntactic structure.[7] That the entity in the second argument of the predicate is the name of a formula but not the formula itself is necessary, since in first order logic no predicate directly over formulas can be introduced.

The most important advantage of this approach, comparing to the modal epistemic logical formalism, is its powerful expressivity. Statements such as "a formula is known by everybody" can be expressed ($\exists y \forall x Know(x, y)$), where in a sense we have quantifiers ranging over formulas. But the disadvantages of this approach are hard to overlook. For one thing is that complications quickly rose when layers of nested epistemic statements are concerned,[8] and for the other, if we incorporate some most agreeable epistemic axioms

[6]For instance, playing chess and proving mathematical theorems, which involve logical reasoning, are typical examples regarded as displays of intelligence. However, this point raises intensive debates within Artificial Intelligent researchers themselves and also between them and philosophers. The debating topics ranging from whether it is possible that machines can have intelligence to whether the linguistic representation and logical reasoning is a sufficient or necessary condition for intelligence. See McDermott (1987); Nilsson (1991); Birnbaum (1991); Brooks (1991); Crevier (1993); Russell and Norvig (2009) for the discussions within the circle of Artificial Intelligence, and Dreyfus (1979); Searle (1980) for philosophers's objections to the idea of Artificial intelligent.

[7]Familiar examples of these theories are elementary number theories such as Peano Arithmetic, or its finitely axiomatized fragment Robinson's system Q (see Tarski et al. (1953)) in which formulas can be coded by one way or another Gödel numbering, and hence can be reasoning within the system.

[8]Statements such as "John knows Mary knows some formula" can't be simply formulated as $\exists x Know(John, \ulcorner Know(Mary, x)\urcorner)$, where x is not a free variable, even plainly a variable, in $\ulcorner Know(Mary, x)\urcorner$.

governing the behavior of these epistemic predicates, we will end up with an inconsistent theory.[9] One direction of epistemic logical research is searching for the consistent fragment of first order theory with epistemic axioms.[10] But the modal approach of epistemic logic still gains the popularity; especially, as we will see, the propositional modal epistemic logic is rich enough to have interesting and useful applications.

It has become a normal practice to begin an illustration of applications of modal epistemic logic by showing that the logical framework can help to analyze interesting epistemic puzzles; and we are not an exception here. Other epistemic puzzles have been analyzed as well;[11] however, we will still investigate the most studied "Muddy Children Puzzle," of which the analysis can be elegantly carried out with the model-theoretical method.[12] Notice that our purpose is not to elaborate the technical details of the formal model, which can be found in most textbooks or introductory articles of epistemic logic. Here we are trying to explicate how the philosophers' abstract creation, possible world semantics, can be utilized to cope with problems derived from practical concerns.

The puzzle has many different variations. We will proceed our discussion based on the following story: A father tells three children in a room that at least one of them has mud on his forehead. No child is able to see their own foreheads but they all can see each other's. Then the father repeatedly asks the children if they know that there is mud on their foreheads. No child answers the question positively in the first two rounds, but at the third round

[9]See Montague and Kaplan (1960); Thomason (1980). Montague and Kaplan proves that adding some intuitive epistemic axioms about knowledge, including $Know(a, \ulcorner\phi\urcorner) \to \phi$ (with our notation), a modal T-like axiom among others, into elementary number theory, the resulting theory is inconsistent. Thomason shows that even without adding the T-like axiom but with some other axioms which are naturally about belief, the resulting theory is still problematic. It turns out that the agent who believes all the axioms of the elementary number theory believes every formula. Montague and Kaplan named the paradox the *Knower Paradox*.

[10]E.g. Perlis (1985) and d. Rivières and Levesque (1988).

[11]Conway paradox (Conway et al., 1977; van Emde Boas et al., 1980) and the puzzle of "Sum and Product" (Freudenthal, 1969) are other instances of epistemic puzzles studied and formulated with possible world semantics (see Parikh (1989) and McCarthy (1978) respectively), where the later is also investigated within the Dynamic Epistemic Logic frameworks (e.g. Plaza (2007) and v. Ditmarsch et al. (2008)).

[12]The same puzzle has been presented in several different forms. One is the version of "Colored Hats" (Littlewood, 1953), and the other the "Cheating Wives" (or Husbands) (Gamow and Stern, 1958; Moses et al., 1986). The version of "Muddy Children" is introduced in Barwise (1981). Analyses of these puzzles within the epistemic logic framework can be found in Fagin et al. (1995b), Osborne and Rubinstein (1994) and v. Ditmarsch et al. (2008).

everyone gives the correct answer simultaneously. The puzzle is to ask how many muddy children are in the room, and the answer is that all of them are.

Of course some preliminary assumptions are needed. Every child should be intelligent enough to perform basic reasoning, and they are truthfully answering father's questions. They all have good visions which make them able to see others' foreheads and also notice that all the children can see each other's forehead as well. But the most interesting part of the puzzle is that children get useful information from other children's ignorance (no one answers the father's question in the first two rounds). This puzzle is not particularly challenging, and it is entertaining to solve it with our informal reasoning. However, possible world semantics gives us a systematic way to solve the problem, and help to establish the logical structure of the puzzle. We have roughly spoken of the possible world semantics for a single reasoner. But now we need a semantics that can model more than one knower. It is actually quite straightforward to generalize the ordinary possible world semantics to concern multiple agents. All we need to do is to supply alternativeness relation for each agent in discussion. Two worlds which are indistinguishable from someone's point of view are not necessarily indistinguishable from the others'. Recall that in possible world semantics an agent a is said to know a proposition ϕ in a world, namely, $K_a\phi$ is true in a world, if and only if the proposition ϕ is the case in all the alternatives, relative to the agent a, to the world.

In order to apply possible world semantics to resolve problems in realistic scenarios, the first thing to do is to decide the totality of the worlds that are relevant to the questions. In our puzzle, they are all the possible conditions about the children's foreheads. Each child is either with or without mud on his forehead. Thus overall we have eight relevant possible worlds. There are many ways to represent these worlds. Let's use triples of 0's and 1's with 0 meaning no mud on the forehead, and 1 otherwise. For example, $(0, 1, 0)$ denotes the world in which only the second child has mud on his forehead. By this way of representation, we also set the truth value of the basic facts in each world. Let p_i for $i = 1$, 2, or 3 denote the fact that the i-th child has mud on his forehead. Then p_2 is true in the world $(0, 1, 0)$, while p_1 and p_3 are not.

The next step is to decide the alternativeness relation for each child. The goal of the alternativeness relation is to connect each world to worlds that the knower can't distinguish from the world that he actually stands on, namely, to worlds that are still possible to the knower, given the information that can be had. In our case here, the making of the decision should be of no difficulty. The only thing that a child can't distinguish is whether there is mud on his forehead. For example, in the world $(0, 1, 0)$, the first child can't distinguish if he is in $(0, 1, 0)$ or $(1, 1, 0)$, and for the second child both worlds $(0, 1, 0)$ and $(0, 0, 0)$ are possible to him. Following the lead of this example, the alternativeness relation for each child can be naturally built. Then putting all the above semantic considerations together, we have the basic formal epistemic structure of the story.

By this setting the alternative relation for every child is the equivalent relation. It is reflexive, transitive and symmetric.[13] That is to say, these children are supposed to be S5 knowers, who know what they know and know what they don't know. This conforms to the description of the puzzle that these children are able to faithfully express that they don't know they are muddy, when they don't. Syntactically, we can increase our understanding of these children's epistemic behaviors by deriving theorems from the axiomatic proof system of multi-modal logic S5n (n is used to denote the number of the knowers, so the modalities, in the system; in our case it is three), where the above semantic construction is a model. However, as we can see in the following, the semantic analysis can give us much insight.

Now the father announces that there is at least one child with mud on his forehead. By this information, we can remove the world $(0, 0, 0)$ from the semantic model in the reasoning that every child hears father's announcement and hence recognizes that the world is impossible. Since at the first round, there is no child answering the father's question, we can remove the worlds which have only one 1 in the triple, i.e., the worlds in which only one child has mud on his forehead. The informal reasoning of this move is that the only child with mud on his forehand can make out that he himself is muddy, since he can see that the other two have no mud on their foreheads. By the formal semantics, we can see that, since the world $(0, 0, 0)$ has been removed, then these triples with only one 1's has no alternative world other than themselves. Hence if the child is in one of these worlds he knows he is in the world and knows there is mud on his forehead.

A similar argument can be carried out to remove worlds with two 1's in the triple when no one in the second round answers father's question. In these worlds the children with mud should know that they have mud right after the first round of the father's question, since the worlds with only one 1 have been removed, and hence no alternative worlds are for these children. As a result, there is only one world left, that is, $(1, 1, 1)$, and so we know what the answer is.

This is a quite simple puzzle, but the analysis of it gives us an idea

[13] For example, $(0, 1, 0)$ and $(1, 1, 0)$ are alternative to each other for the first child, and no others are alternative to these worlds.

of how possible world semantics can help to settle epistemic problems concerning multiagents, who can communicate with each other, even by their ignorance. The puzzle can be generalized to deal with more than three children in the room, and given the systematicity of our construction of possible world semantics for the puzzle, there is no problem to extend our argument to demonstrate the results of these generalized versions of the puzzle. Accordingly, the father should ask as many questions as the number of muddy children in the room in order to have at least one child positively and correctly answering his question.

3.2 Applications for Distributed Systems

Now we discuss a genuine application of epistemic logic. We will explicate how possible world semantics can help to analyze *distributed systems*, an application which is commonly regarded as the most successful of the modal approach of epistemic logic. At the first sight, regarding the remoteness between these two subjects, there is probably no clue as to how this application will take place. Some background will be in order before we discuss further.

A computer program basically is a sequence of instructions written by a language that computers can recognize and then execute. But it is usually not immediately clear for us, the program designers, to know whether a program will carry out the task that it is supposed to do. So it has been suggested to use formal formalisms such as logic to reason about the behaviors of programs.[14] For this purpose, Pnueli's Temporal Logic and Pratt's Dynamic Logic are two notable examples.[15] Pnueli's Temporal Logic has temporal modality semantically ranging over states of the transition in a program, and Pratt's Dynamic Logic opens a new genre of modal logic, which also influences the later development of Dynamic Epistemic Logic. Formulas of the form $[\alpha]\phi$ are introduced, meaning that after the execution of the program α, the statement ϕ will be the case, where the modalities are indexed by structural entities α built up from atomic programs and basic program operations such as composition, choice, iteration and test. With these formalisms, then the requirements of programs can be formally specified and algorithmically verified.

Certainly, a similar formalism is also in need for programs running on a

[14]See the suggestion in Floyd (1967). If the readers don't have programming experience but have the idea of Turning machines, consider the tuples of a Turning Machine which are designed to add two input numbers, represented as binary strings. Without cautious examination of the details, it is not immediately clear that the setting can do its job.

[15]See Pnueli (1977) and Pratt (1976).

distributed system, a system that is composed of a set of autonomous processes, computers or microprocessors, that behave independently and have their own local states for storing the information they have. No process in a distributed system can access the local states of the others; thus no process has the global picture of the whole system; nonetheless, they can aggregate information by sending and receiving messages from one another, through the channels that connect them. But one important feature that distinguishes these distributed systems from others is that for algorithms, usually called *protocols* in this context, running on these systems some uncertainty have to be taken into account. For example, it can't be taken for granted that the connecting channel is reliable, and hence that a sent message can always reach its destination; and it is not guaranteed for a process that the other processes that it communicates with are functioning properly, and hence always deliver correct messages. As a result, the aforementioned formalisms are not adequate, and new formalisms are in demand.

To illustrate the distinctive task of the design of algorithms, consider the following fundamental scenario in distributed systems. Given two processes, one sends out a message in its local state to the other and then wants to know if the message arrives at its destination. Even for this simple requirement, a distributed algorithm is necessary, since the message could be lost in the traveling. However, from our daily experience, a protocol for this purpose is not difficult to conceive. We can require that the receiver sends back a message of acknowledgement once it receives the message from the sender, and the sender, on the other hand, keeps sending out the same message, in case the receiver doesn't receive the message, until it receives an acknowledgement from the receiver. Our informal wisdom tells us that this protocol surely works, since after receiving the acknowledgement, the sender has enough information to know that the message has safely reached its destination.

Now, we need a formalism which can reason about distributed algorithms and hence, as a test, should have enough machinery to formally describe and prove that the above mentioned small protocol can do its job. An immediate proposal might be to come up with a formalism which has the expressivity to explicitly express the details of state transitions of processes in distributed systems. However, our informal description of the above and other protocols have given us a hint of what a more simplified and intuitive formalism can be. We attribute mental qualities to processes, describing them as having knowledge. Although, as we understand, it is just a way of speaking without supposing that processes are capable of having any knowledge in the ordinary sense, the idea of using formal epistemic models to analyze distributed algorithms has suggested itself. Modal epistemic logic and possible world semantics become relevant.

To construct a possible world model, as we've discussed, we need to determine the totality of the possible worlds. When a protocol runs on a distributed system, the local states of processes will change over time according to the protocol. Although for each process, there is no complete information of the whole system at each time, we as an outsider can depict the snapshot of the system by the collection of the local states of the processes of each time. We will call these snapshots as the *global states* of the system, and for convenience, a state of the environment, which describes information like a message being lost or not, can also be included in a global state. Then the global states that are reachable when a protocol is executed are all the possible worlds we need for the construction of the epistemic semantics for the protocol. The reachability of a global states can be defined as being an element in a sequence of global states called a *run* of the protocol which begins with a possible starting global state and for any two consecutive global states, processes change their local states accordingly, or the environment varies without any change of local states.

The alternativeness relations of these semantic models can be determined in an intuitive way. For each process, which is our knower, in a possible world, the only information it can has is the information in its local state, and hence it can't distinguish worlds in which the local states of the process are the same. These worlds are alternative to each other with respect to the process. Consequently, the alternativeness relation for each process is the equivalence relation, and every model constructed in this way is an S5 model.[16] In a sense, all the processes in distributed systems are considered as an S5 knower, and hence their epistemic behavior is described by the modal epistemic logic S5.

Having the general idea of simulating the behavior of distributed algorithms, we can come out with an epistemic model specific for the aforementioned simple protocol about message transferring, and then formally prove the correctness of the protocol. Roughly speaking, it can be shown in the established model that for all the possible worlds in which the local state of the sender is in the state of "acknowledgement received," the fact that the receiver has got the message is the case, and hence the sender knows that

[16] There are many possible world semantic proposals for distributed systems around the eighties of the last century. The version we describe above is basically following the one presented in *Reasoning About Knowledge* (Fagin et al., 1995b). In this version how a process evolves from the start into its current local state is irrelevant; but it is reasonable to have an epistemic model such that the processes are sensible to their "histories" in the midst of the execution of the protocol. An example of such is Parikh and Ramanujam (1985). For other epistemic models for distributed system, see Chandy and Misra (1986), Fischer and Immerman (1986), and Ladner and Reif (1986).

the sent message has been received.

Hintikka once put it that the idea behind the possible world semantics for epistemic notions is "the old adage *information means elimination of uncertainty*,"[17] and, as we have seen, the difficulty arising in the modeling of distributed systems is exactly to find a way to deal with the uncertainty. Probably possible world semantics is not only just one of the formal models for dealing with distributed systems, but it exactly captures the meaning of the word "knowledge" when we verbally attribute it to processes.

Part of the reason for the success of this application of epistemic logic is that the logical omniscience problem does not cause trouble. We can let a process knows all the logical consequences of its knowledge without a problem insofar as this information can help us, protocol designers, to understand better the algorithm, and then improve the protocol. This is exactly the purpose when McCarthy suggested to ascribe mental qualities to machine.[18] Nevertheless, when we apply this modal approach of epistemic logic to intelligent machines for which we assume they can generate their own knowledge and act accordingly, the unrealistic reasoning ability suggested by modal epistemic logic appears to be inadequate. So more delicate considerations for these cases are necessary.

3.3 Common Knowledge

Right now we have an extended version of modal epistemic logic which can reason about knowledge of multiple knowers in which each knower is associated with its own alternativeness relation. Roughly speaking, this framework is a collection of independent copies of modal epistemic logic for individual knower;[19] no new epistemic concepts relative to a group of knowers are introduced. But in our daily life such concepts often play an important role. For examples, if a couple lost each other in a department store of several floors, they usually can find each other at the same floor. A good explanation for this phenomenon is that, except in the case of pure luck, the couple have common knowledge of where they both usually like to be, like a coffee shop. They both know their partners like to be there. But this is not enough for

[17] Hintikka (1986).

[18] Here's the first sentences in McCarthy (1979): "To ascribe [mental qualities] to a machine or computer program is legitimate when such an ascription expresses the same information about the machine that it expresses a person. It is useful when the ascription helps us understand the structure of the machine, its past or future behavior, or how to repair or improve it."

[19] This is clearer from the syntactic point of view. The axiom system S5n is actually a collection of S5 modal logics of which each captures the epistemic properties of each agent.

them to meet each other. They also have to know that the other knows that they know they both like the place. Otherwise, it can't be sure that their other half will show up at the place if they don't know their partners know it. This argument can go on forever. They also both know that the other knows that they know that they like the place, and so forth. So it is always worthwhile for the one coming to the place earlier to wait there but not to move around trying to find the other one.

The other example is that when we leave a theater after watching a movie, we usually will ask our friends "how was the movie?" and then a conversation starts. But again, how can this be possible? since in the question there is no mention of which movie we are talking about. However, of course, what is the movie in question is common knowledge between us and our friends after just leaving a theater. We know that they know what we are talking about, and also know that they know we know they know what we are talking about since only then they can answer the question without doubt that we understand what they are saying. And again the argument can go on to show that this chat can take place due to the fact that we know that our friends know that we know that they know which movie we are talking about, and so forth.

So an event which is commonly known by a group of knowers is full of interest, and it is also directly related to the areas where epistemic logic is applicable. Going back to the puzzle of muddy children, careful readers might find that the content of father's announcement is actually nothing new to the kids. Given that all children have mud on their foreheads, everybody knew before the announcement the fact that at least one of the boys was a muddy child in the room. But what makes the announcement so special is that now children have an idea about the knowledge of the others. They now know that other children also know the fact they know, but furthermore, they know that everybody knows that everybody knows that, and so forth. That is, father's announcement makes the fact become *common knowledge*. As we've seen, the most interesting part of the puzzle is that children's ignorance can be useful information, but this won't be the case if children have no idea of the other children's knowledge. For example, after nobody answers father's question at the first round, children can't therefore infer that there is at most one child without mud in the room, since without knowing that the other children also know that at least one child has mud on the forehead, children can't expect that the other children are able to conclude whether they themselves are muddy or not.

The common knowledge of a formula ϕ is usually denoted as $C\phi$, and a relative concept, mutual knowledge, that everybody knows, is denoted as $E\phi$. $E^k\phi$ will be the shorthand for $E\cdots E\phi$, with E repeating k times. Informally, as what we've discussed above, that a proposition is common knowledge just means that everybody knows the proposition, everybody knows that everybody knows that, and so forth, which can be represented by the infinite conjunction of all levels of mutual knowledge $E\phi \wedge EE\phi \wedge \ldots \wedge E^i\phi \wedge \ldots$; but generally speaking, this can't be formalized in a regular modal logical setting, where formulas with infinite length are illegitimate; instead, some fixed point schema are employed to syntactically categorize common knowledge.[20] Semantically, however, the informal idea of common knowledge can be quite well captured. That a fact ϕ is mutual knowledge $E\phi$ in a world w means the fact is true in every world w' which is indistinguishable by either one of the knowers in the group in discussion. We call these worlds w' the first-level mutual worlds of w, and we call worlds the second-level mutual worlds of w if they are the first-level mutual worlds of the first-level mutual worlds of w. We can continue to define any level of mutual worlds by the similar method. Then a fact ϕ is mutual knowledge of the second level in a world w, namely, $E^2\phi$ is true at w, if and only if it is true at all the worlds w' which are either the first-level mutual worlds or the second-level mutual worlds of w, and a fact ϕ is common knowledge $C\phi$ if and only if it is true at all mutual worlds of all levels.[21]

Now adding common knowledge into the multimodal epistemic logical formalism, we have the machinery to specify statements involving common knowledge, and then prove these statements. Taking the machinery to reason the muddy children puzzle, we can then formally prove that if children at the end can't figure out their own status of muddiness, the fact, call it ϕ, that at least one child is muddy in the room is not common knowledge. Formula $\sim C\phi$ can be formally justified. However, this is not a *tight* result. More cautious readers might find that in this case, reasoning about the muddy children puzzle, common knowledge in its full strength is not necessary. Children need not to have common knowledge in order to figure out they have mud on their own foreheads. We only need ϕ to be mutual knowledge up to the level equal to the number of muddy children in the room. In our example, it

[20]There are two different axiomatizations of common knowledge in the literature. See Fagin et al. (1995b) and Meyer and Hoek (1995), respectively.

[21]The definition of common knowledge within the modal logical framework that we give here is the one commonly in use these days. In the literature, the first trial of formulating common knowledge in the possible world semantics is by "any fool O" in McCarthy et al. (1978) (if the fool knows, then everybody knows ($O\phi \to K_i\phi$)). The concept of "any fool" is actually a weaker notion of common knowledge, $O\phi \to C\phi$, and equivalent to the concept of "justified common knowledge" introduced in Artemov (2006). See Antonakos (2007). Aumann formulates the common knowledge in (1976) with event-based models, which we will discuss later. Also see Barwise (1988) for comparisons of different definitions of common knowledge.

is $E^3\phi$.[22] But this doesn't diminish the applicability of common knowledge. As we should see, the concept of common knowledge with full strength is irreplaceable for the study of distributed algorithms, especially employed to show some negative results.[23]

One folklore problem in the study of distributed systems is to ask whether it is possible to construct a protocol such that two processes can work out a time to act together simultaneously through finitely exchanges over an unreliable connecting channel. This is the famed "Coordinated Attack" problem, originally dramatically presented as a story about two generals on the tops of two hills trying to attack the enemy in the common valley. The chance that they can beat the enemy is that they attack simultaneously, but the only way they can communicate to each other is by sending messengers running through the valley with the risk to be captured. So the question is, is there a protocol such that the generals can apply to determine a simultaneous attacking time?[24]

Informally we can quickly conclude that it is impossible since no matter when a process prepares to accept a time for action, it still has to worry that the other side doesn't get its previous message and hence won't act at the accepted time. However, to extinguish any hope that there might be some elegant way of dealing with the problem out there but we just don't know it, a formal proof to confirm this impossibility result has been urged.[25] We have shown that the computation of any protocol running on distributed systems can be simulated by possible world semantics, and hence modal epistemic logic with common knowledge is a natural choice.[26] With such a formalism, it has been shown that if two processes can reach consensus about the time of their actions, then the message of the time must be common knowledge. But, on the other hand, unfortunately, it is also shown that under the circumstances where, in some formal sense, the safety of the communication

[22] That is, when $E^3\phi$ is the case, our previous argument in Section 3.1 will still hold. This however can be achieved by that the father doesn't speak publicly the fact that there is a muddy child in the room, but speak independently, without the listening of the others, to each child the fact, and then that he told the other children the fact, and then speak independently to each child that he told the other children that he told the other children the fact.

[23] For the early discussions of adding common knowledge into modal epistemic logical systems for the purpose of reasoning about distributed algorithms, see Lehmann (1984); Fischer and Immerman (1986); Halpern and Moses (1990).

[24] See Gray (1978). One similar problem, the electric mail game, is studied in the context of game theory Rubinstein (1989).

[25] The need of formal proof is discussed in Yemini and Cohen (1979).

[26] Halpern and Moses (1990) is the first to discuss the coordinated attack problem within the modal epistemic logical framework with common knowledge.

channel is not guaranteed, common knowledge is not obtainable through finitely many communications.

The analysis of common knowledge is also where the studies of different disciplines meet. The concept was first introduced in Philosophy by Lewis,[27] studying the coordinate problem, of which the department store story is an example; and then define *convention* in terms of common knowledge of regularities in behavior of members among a population, aiming at answering Quine's challenge of analyticity.[28] Common knowledge is also studied in Psychology of language comprehension, as shown in the movie-chatting example,[29] and now we see it in Computer Science and Artificial Intelligence. Finally, it is a central concept in Game Theory as well, where common knowledge of players' rationality, and the rule of the game are often the basic presumptions for the analyses of games to proceed.

The first formulation of knowledge and common knowledge in the study of Game Theory is due to Aumann (1976). In this seminal work, Aumannn nonetheless shows that, as explicitly stated in the title, we can't "agree to disagree." Speaking more formally, he shows that if players have the same ideas of the world, manifested by their prior beliefs, i.e., distribution functions assigning probabilities to the relevant states of the world, then even if players later receive different information, as long as their posteriors, the distribution functions after the receiving of the information, are common knowledge (agreement), the posteriors will be the same. This result is surprising since in our everyday life we do "agree to disagree" on many things; but what is emphasized by Aumann is that if players in a game recognize each other's rationality and intelligence and hence respect each other's opinions, then the different posteriors just shows that their opponents have genuine information that is different from theirs. Taking this into account, revising their own posteriors, eventually players will come to the same idea of the world.[30]

Aumann's formal treatment of epitemic concepts has its origin in the

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[27] Lewis (1969).

[28] Quine (1951).

[29] See Clark and Marshall (1981); Clark (1992). In this context, common knowledge between two persons in a conversation is used to be called mutual knowledge.

[30] Here's a quote from Aumann (1976), where subjective probabilities are players' posteriors: "reconciling subjective probabilities makes sense if it is a question of implicitly exchanging informatin, but not if we are talking about 'innate' differences in priors."

Following Aumann's result, some extensions are studied. The updating of the probability of the world until common knowledge is reached is formally shown in Geanaokoplos and Polemarchakis (1982). However in Parikh and Krasucki (1990) it is proved for a group of more than two players, if they communicate in pairs then consensus can be made without common knowledge.

probability theory. An event E is defined as a subset of a set of states of nature Ω (as "probability space" or "sample space" in probability theory), and each player i will be associated with a partition $\mathcal{P}_i$ on Ω, in which an element $P_i(w)$ means that the information that the player is informed at the state w in Ω. Then the event that player i knows the event E, K_iE, is defined as the subset of Ω which contains states w such that $P_i(w) \subseteq E$, that is, K_iE is the collection of all the states w such that according to the information that the player has at w, event E is the case. This formulation is different from our familiar setting of possible world semantics; but readers who know a little of discrete mathematics can immediately realize that it is mathematically equivalent to the possible world semantics for S5. Given a set of elements, equivalence relations and partitions can define each other on the set. Thus, Aumann's definition of common knowledge is equivalent to what is defined in possible world semantics, but in the language of partition.[31]

The same epistemic logical framework is also under different interpretation in game theory, in order to be employed to study problems in a larger picture, under the head of *interactive epistemology*.[32] As in the study of agreement theorem, a space of the states of nature is taken as a description of relevant possibilities with respect to the players' or decision-makers' points of view of the worlds. However, there are issues in game theory such as players' knowledge about the other players' rationalities that can't be modeled. To do so, we need a state to have a full description of the nature of the game from an outside observers' point of view which thus includes such as players' actions chosen at the state, their beliefs of the actions chosen by other players at the states, and their beliefs of the world. One important application of this interpretation is that then we can investigate that what kind of epistemic assumptions on the players will lead these rational players to adopt the stately suggested by the proposed solution concepts of games, such as Nash equilibrium. For example, it is usually implicitly assumed that if the players are rational and have common knowledge of the rationalities and actions of the other players, then Nash equilibrium will be played. Nonetheless, a formal model can be applied to show that in the case of pure-strategy two-person games in which there is only one Nash equilibrium this condition is too strong. If the players, as it is shown, are rational and their actions are mutually known, then the choices of their actions just constitute Nash

[31] According to Aumann (1976), event E is common knowledge at w between two players with partitions $\mathcal{P}_1$ and $\mathcal{P}_2$, if it includes the member of *the finest common coarsening of* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *that contains* w, which can be proved to be equal to the set of worlds that are mutual worlds of w of finite level with respect to the alternativeness relations corresponding to $\mathcal{P}_1$ and $\mathcal{P}_2$.

[32] See introduction in Aumann (1999, 1998).

equilibrium.[33]

3.4 Dynamic Epistemic Logic

The modal approach of epistemic logic is now the standard formalism for reasoning about knowledge and belief. Recent developments seem to echo Scott's once advice that we should include more than one modal operator in one modal system.[34] The suggestion is not about introducing systems with multiple agents, but implying systems of combining different modal operators standing for different concepts. One of the most studied of such logical frameworks that includes epistemic modalities is Temporal Epistemic Logic, which has been introduced for the formal verification of distributed algorithms with expressivity richer than that of simple epistemic logic. From the semantic point of view, this development is not unexpected; it is obvious that there are temporal elements embedded in concepts such as a *run*, and hence temporal modalities can be naturally introduced to semantically range over these elements. In the literature different temporal logical systems have been put forward for reasoning about linear-time or branching-time semantics,[35] and therefore various temporal epistemic logical systems have been proposed,[36] with different combinations of temporal systems and epistemic logic and with the introduction of different connection axioms between epistemic and temporal modal operators.[37]

The subject of Artificial Intelligence at its birth was set to build a machine or computer program with general intelligence, capable of solving problems in a broad range; but sooner or later, in practice, the research has shifted

[33] For the epistemic conditions of the other solution concepts, see discussions in Aumann (1987), Brandenburger (1992), Aumann and Brandenburger (1995).

[34] Scott (1970).

[35] Some well-studied temporal logics, especially for the purpose of formal verification, are LTL, Linear Temporal Logic (Pnueli (1977)), CTL, Computational Tree Logic (Clarke and Emerson (1981)), and CTL* (Emerson and Halpern (1986)).

[36] Several earlier introduced epistemic logical systems for reasoning about distributed algorithms includes temporal modalities (Parikh and Ramanujam (1985); Ladner and Reif (1986); Lehmann (1984); Kraus and Lehmann (1988)).

In Halpern and Vardi (1989, 1988), a systematic model-theoretical study of the combinations of epistemic and temporal logical systems are given, and their complexity results are studied; the complete axiom systems for these models can be found in Halpern et al. (2004) and van der Meyden and shu Wong (2003).

[37] One example which is trying to capture the semantic idea of *perfect recall* (what has happened is always remembered) is $K_a\Box\phi\rightarrow\Box K_a\phi$, where $\Box\psi$ means that ψ is always true; but it has been shown that more complicated axiom schema is needed for the task. See discussions in Halpern et al. (2004).

to concentrate on more practical and specific-purpose applications, of which theorem proving and natural language processing are examples. Unsatisfied with this development, researchers started to urge the study of integrated agents who are supposed to be able to carry out plans to resolve diverse tasks that they are assigned. It has been argued that in order to characterize these autonomous agents, we need to attribute mental qualities to these agents more than just belief.[38] For this purpose, we have seen a logical framework for reasoning about agents including belief, desire (or goal), and other modalities in which intention is defined as choice of commitment; and a model of rational agents in which all belief, desire and intention modalities are defined on a branching-time possible world semantics.[39]

Before closing the discussions on the applications of epistemic logic, we will review a new trend of study which in many perspectives continues the development of the modal approach of epistemic logic. Going back to the muddy children example once again, one thing not formulated inside a logical framework is the activity of announcing, either by father or by children with their silence. Recall that in our argument we use a possible world model to approach the answer. For each announcement, we delete some worlds which can't be the cases based on the announcement. This transformation from model to model is not part of any logical framework we have discussed so far, but it seems plausible to propose such a semantic structure to reason about these announcements which cause the shrinking of the models. So to speak, in the proposed semantics, to check if a formula ψ is true at a world w of a model M after a formula ϕ is announced to be true, is to check if ψ is true at w in the submodel of M in which all the worlds where ϕ is false are eliminated. And this is exactly the motivation of the logical framework Logic of Public Announcement to be introduced, in which new formulas such as $[\phi]\psi$ are suggested with the intended meaning that after the announcing of ϕ, ψ is true.[40]

With this formalism, we have the machinery to formulate such sequences of events in one formula. For example we can formally formulate such a formula: $[\phi][\psi][\psi]\theta$, where ϕ is father's announcement that at least one child is

[38]See Bratman (1987).

[39]See Cohen and Levesque (1990) and Rao and Georgeff (1991), respectively.

[40]This logical framework is first proposed in Plaza (2007) in which it is the formulas of the form $\phi + \psi$ to be newly introduced, with the intended meaning equivalent to $\sim[\phi]\sim\psi$. The formalism that we see here, however, is independently introduced in Gerbrandy and Groeneveld (1997), with a more general semantics where after the announcement, which is believed to be true by the listeners, the alternativeness relations are restricted to those worlds in which the announcement is true. Adding common knowledge into logic of public announcement is first studied in Baltag et al. (1998).

mud, ψ denotes no child knows if they are muddy, and θ means that everybody knows that they are muddy. Then with the just mentioned semantics we can prove that this formula is true at the world $(1, 1, 1)$ in the model established for the puzzle of muddy children. We can also prove that, with a similar argument, the formula $[\phi][\psi]\theta'$, where θ' means the second child knows he is muddy, is true at the world $(0, 1, 1)$. It is obvious that the dynamic flavor is certainly the distinguished feature of this semantic setting; and now Logic of Public Announcement is one of the members in the family of Dynamic Epistemic Logic, in which more logical frameworks with dynamic semantics are suggested.[41]

One special phenomenon of Logic of Public Announcement is definitely worth some words about it. Unlike most commonly seen logical frameworks, the rule of uniform substitution is not applicable in the Logic of Public Announcement. Consider formulas of the form $[\phi]\phi$, which means that after ϕ's announcement, ϕ is still true. It turns out that not all formulas of this tautology-like form are tautologies. When ϕ is a simple fact about the world, that is, not about the epistemic state of the knower, then according to the dynamic semantics, $[\phi]\phi$ is valid, true in all worlds in all models. Actually, $[\phi]K\phi$ is also valid since all the worlds in which ϕ is false will be eliminated. So even if originally in a model in a world, in which the knower in discussion doesn't know the fact ϕ, after the announcement he learns something, and this result is probably what we would like to have when this dynamic semantics is introduced. But a problem is also formed because of this learning process. Since we also can announce formulas concerning the epistemic state of the knower, if the formula $\phi \wedge {\sim}K\phi$, that ϕ is true and the knower doesn't know ϕ, is truthfully announced, then the formula $\phi \wedge {\sim}K\phi$ cannot continue to be true after its own announcement, since the knower now comes to know ϕ from the first conjunct of the conjunction. So we see the Moore sentence causes a problem once again. In the literature, if $[\phi]\phi$ is valid, ϕ is called a successful formula. Equivalently, ϕ is successful if and only if after its announcement, it becomes common knowledge. So Moore sentence is a counter instance of successful formulas, and technical properties related to successful formulas are a topic for further studies.[42]

In the literature now we can see Dynamic Epistemic Logic is taken to deal with the verification principle, and also see the Surprise Examination para-

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[41] Examples of other dynamic epistemic logics are epistemic action logic van Ditmarsch (2002) and the action model framework Baltag et al. (1998). See more general discussions of these logics in the textbook *Dynamic Epistemic Logic* (v. Ditmarsch et al., 2008) written for this subject.

[42] See Van Ditmarsch and Kooi (2006), van Benthem (2003) and v. Ditmarsch et al. (2008).

dox being scrutinized in the logical framework, where the teacher's "surprise exam" announcement is analyzed as an unsuccessful formula.[43] So epistemic logic is taken to settle philosophical puzzles once again. This makes the development of epistemic logic up to this point back to its original intent when it was introduced. Or we should say this is a new developing phase of epistemic logic, where the meaning of epistemic notions is understood from the changes of epistemic states.

3.5 Some Logically Non-Omniscient Systems

So far we have seen successful applications of modal epistemic logic with the semantics of possible worlds. But it is still easy to come up with examples which are supposed to be analyzed by formal epistemic methods, but epistemic logical systems with the logical omniscience problem are certainly inadequate. One of the folk examples is the analysis of chess game. Given that the players are fully aware of the rules of the game, if they are logically omniscient and hence can always figure out the best moves for their opponents and themselves, then the game should never start, since from the beginning the players know it is White's game, Black's game, or a draw, a consequence completely contrasting with what happens in reality. Another example is in the analysis of cryptography, of which the goal is to hide information such that only the intended group of people can disclose the hidden information. In an epoch of computers like today, information is digitalized and distributed publicly, so the challenge for cryptographers is to devise a mathematical algorithm such that even if people not in the intended group have the encrypted information in public and an idea of the encryption algorithm, they are still unable to reveal the true information in the period of time worth devoting to decrypting it. However, unfortunately, given that the basic axioms of mathematics are known to people, a logically omniscient agent is always able to disclose the hidden message in an instant of time.

So for researches there are enough reasons to set up new approaches towards epistemic logic such that the problem of logical omniscience doesn't occur. Many alternative approaches for this purpose has been proposed. Their technical details can be found in their respective references or some survey articles; hence the efforts won't be repeated here.[44] Instead, we will discuss some of the common strategies employed by these approaches, so as to give a general idea of the practice of researchers when the logical omniscience

[43] Van Benthem (2004) and Gerbrandy (2007).

[44] See Chapter 9 of Fagin et al.'s textbook (1995b), and Moreno (1998) for a more comprehensive survey.

problem is the main concern. We will also discuss their respective deficiencies. None of these approaches are perfect.

It is nature to blame that the root of the problem of logical omniscience rests on the setting of possible world semantics. In each world, the set of true formulas is closed under all the classical logical rules, so immediately it follows that the modeled agent knows all the logical true statements and logical consequences of his knowledge. A radical response to this analysis is to give up the semantics thoroughly, and instead, simply consider knowledge or belief of an agent as a set of formulas.[45] This approach gives us a lot of flexibility to model agents' epistemic attitudes. Any set of formulas, even inconsistent one, can form a belief set of an agent, not to mention a set of formulas which is not closed under logical consequence. However, the problem of this approach is also in this flexibility. It is so general that we can't derive any property of the knowledge or belief possessed by the modeled agents. Agents simulated by these models are not assumed with any logical reasoning ability. Thus to create a belief set is to know all the agent's beliefs in advance, and to decide whether a formula is believed is to investigate the whole belief set in order to see whether the formula in discussion is in the set. As it is commented, this approach gives us a way of "*representing* knowledge rather than *modeling* knowledge."[46] In other words, it is hard to regard these models as logics of epistemic notions.

For an obvious reason the strategy behind this approach is called syntactical. A more subtle way to apply this strategy is to categorize an agent by a set of base formulas and a set of deduction rules, and then the belief set is the closure of these base formulas under the rules.[47] The reason that these approaches are able to deal with agents without logical omniscience is that there is no pre-requirements for what the set of deduction rules should be. It is not necessary to be complete in the standard classical sense, and, furthermore, as it is noted, the rule can even be in the following form:

$$\frac{D(k) \wedge \phi \qquad D(l) \wedge (\phi \to \psi)}{D(k{+}l{+}1) \wedge \psi}\,, \quad \text{for } k{+}l{+}1 \leq n,$$

where, e.g., $D(k)$ means ϕ is derived through k times of applications of logical rules from the base formulas.[48] That is, the agent is supposed to be able to

[45] E.g. Moore and Hendrix (1979).

[46] See Fagin et al. (1995b).

[47] Approaches of this kind can be traced back to Eberle (1974), followed by Konolige's *Deductive Belief Model*, (1986). Later Wooldridge's *Belief model* is an abstract form among them (1994).

[48] See Konolige (1986). The rule is verbatim from the monograph.

perform *Modus Ponens*, but the times of the rule applications should be limited. Restriction of this kind is actually essential for modeling logically non-omniscient agents in these approaches. Since suppose an agent is able to perform a rule of a general type (no qualification of $D(x)$) without limitation, the reasoning ability of the agent is still logically omniscient. He then knows or believes all the logical consequences with respect to this general rule.

The difficulty of these approaches is to decide the set of deduction rules, especially to decide which of these restricted rules should be included to model an agent. These rules are so artificial that no human agents truly apply deduction rules of such kind. Even for machines or programs designed to be intelligent, it is not realistic to construct them with the ability to execute these rules. The goal of these rules is to model machines or programs which are sensitive to their resources. But it should turn out to be at some point that it is easier to simply perform the rule, taking Modus Ponens as an example, than to check whether their resources have been used up. However, a general complaint about these approaches is that they lose the elegance that the semantic approaches can provide, as we have seen from how some applications can be swimmingly handled by possible world semantics.

So other approaches are proposed to retain the possible world semantics with, of course, the semantics altered in one way or another to conform to our intuition or perspective of epistemic notions. Approaches belonging to this category are many;[49] here we will consider two prominent examples, Montague-Scott's Neighborhood Semantics and the impossible world semantics. The Neighborhood Semantics in a sense is the semantic counterpart of the syntactical approach. The difference is that in the syntactic approach, the objects of mental activities of knowing and believing are taken to be sentences, but in Neighborhood Semantics, they are *propositions* which are equated to sets of possible worlds. Then each world is assigned a set of propositions that are supposed to be known or believed by the agent in the world. The rule of Closure of Logical Consequence is invalidated, and hence agents are not supposed to know all the logical consequence of his knowledge. The advantage of this approach is certainly that it is still built up on the possible world semantics, and the suggestion that the objects of knowledge and belief are propositions has its philosophical grounds. But one of the problems of this approach is that although it escapes from a stronger version of logical omniscience, the modeled agents are still logically omniscient in some way.

[49] E.g. Impossible World Semantics (Rantala, 1982), Nonstandard Structure (Fagin et al., 1995c), Logic of Local Reasoning (Fagin and Halpern, 1988), and Fusion Models (Jaspars, 1991). Some suggestions for using non-classical logic for reasoning about knowledge and belief, e.g. Montague-Scott's Neighborhood Semantics (Montague, 1970; Scott, 1970), can also be included in this group as well.

The rule of Closure of Logical Equivalence: $\dfrac{\vdash \phi \leftrightarrow \psi}{\vdash K\phi \leftrightarrow K\psi}$ still holds in Neighborhood Semantics, and normal agents are still unable to know all the logical equivalences of his knowledge.

In other cases, normal worlds are suggested to be alternative to nonstandard or impossible worlds, worlds in which not all classical logical rules can be applied. The idea behind this semantics is that since normal agents are not perfect reasoners, so in most circumstances they are not able to rule out some worlds where the classical rules cannot be applied. Since agents can't distinguish these impossible worlds from possible ones, they don't know all the logical consequences of their knowledge.

Now the advantage of impossible world semantics is that it retains the aesthetics of semantics, and the design has its intuitive appeal. Furthermore, it has been proved that almost all the other approaches toward logically non-omniscient epistemic logic can be simulated by impossible world semantics.[50] That is to say, the expressivity of the impossible world approach is no less then the others, and hence it is suggested to take impossible world semantics as a general framework for the logical analysis of epistemic notions. But this semantic model faces a practical issue; it has been argued that it is hard to apple the semantics.[51] The difficulty is that it is usually not clear which impossible worlds should be included when a semantic model is established, and how the alternativeness relations between possible worlds and impossible worlds are connected. We are pretty sure of what a possible world should be, but there is a wide range of diverse impossible worlds. So if there is no systematic way of implementing impossible worlds, then the difficulty of constructing a model in such a semantic approach is no less than deciding which formulas should be included to form a belief set, as the syntactic methods demand.

The final approaches we are going to discuss have characteristics from both of the above groups of approaches. It starts from Levesque (1984), which is an influential work in the study of the logical omniscience problem. Following this seminal work, many other logical frameworks has been proposed.[52] To be precise, Levesque's approach should belong to the semantic category. It suggests an adjustment which is purely on semantics. A class of *situations* (partial possible worlds) is introduced to separately support the truth and falsehood of formulas. But the idea of suggesting a system incoporating two concepts of belief: *explicit* and *implicit*, in which implicit belief is

[50]See Wansing (1990).

[51]Halpern and Pucella (2007).

[52]E.g. Lakemeyer (1987, 1991), Delgrande (1995), Thijsse (1996), and also Fagin and Halpern (1988).

the logical consequences of explicit belief, leads to the well-known Fagin and Halpern's awareness approach. The awareness approach greatly simplifies Levesque's setting and allows nested belief statements to be expressed. In the approach, implicit belief is what is justified by the possible world semantics, and explicit belief is a subset of implicit belief which the agent is aware of, and the awareness of an agent is simulated by a syntactical function.

From the above discussions, the semantic approaches retain more of the advantages of the modal approach of epistemic logic, and the syntactic approaches seems to provide more reasonable mechanism to resolve the logical omniscience problem. And we see that this awareness approach inherits the strengths from both of the approaches, and the uncomplicated semantic structure makes it is easy to be applied. However, it faces a new challenge that other approaches won't confront. As Konolige (1986) has argued, the ideology of the approach is problematic, since it is not the case in the real world that "agents compute all logical consequences of their belief, throwing away those not in the awareness set, ..." Analyzing the awareness semantics, it can be found that the possible world part of the approach seems to be doing nothing but providing the validity to formulas to be explicitly believed. But this can be done in the deductive method by enforcing that the belief set is determined by sound inference rules. Then there is not so much difference between the awareness approach and the deductive method.

Chapter 4

Timed Modal Epistemic Logic

4.1 The Problem of Logical Omniscience and the Time of Reasoning

One of the main goals of this thesis is to deliver a logical framework which can serve as a more adequate formalism reasoning about knowledge and belief of normal agents.[1] The modal approach of epistemic logic as we have discussed suffers from the logical omniscience problem. In the previous chapters, we survey the development of epistemic logic, and at the end, some of the commonly studied solutions to the logical omniscience problem are discussed and evaluated. All of these solutions, as we have addressed, have their respective problems to meet. To begin this chapter, however, we will discuss a more general difficulty shared by these approaches towards logically non-omniscient epistemic logic, and then argue that a solution from a completely different perspective is indeed in need to vanish the problem of logical omniscience.

Consider the following rule of Closure of Logical Consequence, which is both syntactically and semantically derivable in MEL, modal epistemic logic, and is the deduction rule that is most debated when the logical omniscience problem is concerned:

$$(1) \qquad \frac{\vdash \phi \to \psi}{\vdash K\phi \to K\psi}\ .$$

It says that if the knower knows the premise of a logically true implication, then the conclusion is also known, which is hardly something that a normal realistic agent can relate to. An agent, as an example, who knows all the axioms of the elementary number theory knows, according to the rule, all the

[1] This chapter is based on the work in Wang (2011a).

theorems of the theory, including Goldbach's conjecture, provided it is true, even though no one in the human history has ever confirmed it yet.

A common and reasonable response to the concern raised by this rule is that an epistemic logical system with such a derived rule permits the knower knowing too much. The modal approach of dealing with epistemic concepts renders a logical framework way too strong to model normal agents, and hence we need something weaker than modal epistemic logic. This analysis of the source of the logical omniscience problem, and the reasoning to resolve the problem is what is taken by most proposed solutions, including what is discussed. According to the analysis, formal epistemic structures are suggested to be established to model agents whose ability to perform logical reasoning is confined, and hence the deduction rules such as the rule of Closure of Logical Consequence can be invalidated. However, as the experience of the practice of researchers reveals and the discussions of the last section's survey shows, we don't have a satisfactory result following this path. We either have an epistemic structure which is general enough to even model agent without any ability to perform logical reasoning, such as in the syntactical approach, but from which no interesting epistemic rules or epistemic properties of the reasoning agent can be derived; or reach an epistemic logic modeling agents with some type of reasoning ability, like in the Neighborhood Semantics, and hence some interesting epistemic properties can be derived, but result in an epistemic structure which is logically omniscient of some kind; for example agents still know all the logical equivalences of their knowledge.

The lesson here is that in our vision, epistemic logic is aiming at revealing the epistemic aspects of rational agents in the sense that they are the beings who can perform logical reasoning, and only in an epistemic modeling of this type of agents we can derive interesting epistemic results concerning the agents; but, on the other hand, we don't want the resulting logical framework fall prey to the logical omniscience problem neither, for if this is the case, the applicability of epistemic logic and the derived epistemic results about the agents are limited. To say an agent is rational is to say that he has the ability to apply logical rules, which is nothing to do with how many times of the applications can be made by the agent. However, the idea of bringing in logics weaker than the modal epistemic logic can't supply a satisfactory solution. These weaker epistemic logical frameworks can't at the same time meet both the goals as described above.

Now let's go back to the analysis of the source of the problem. The reason that a normal agent does not know all the logical consequences of his knowledge is that some consequences are just computationally too far to reach. It is not because normal agents have weaker reasoning mechanisms or lack logical reasoning ability. So in order to solve the logical omniscience problem, the suggestion is to find a way to distinguish relatively easy consequences from difficult ones. However, the expressivity of MEL is too poor to have such information for us to make the distinction. In MEL, and other alternative epistemic logical frameworks discussed, only the content of knowledge is represented, and the temporal information as to when a proposition is known is not explicitly expressed. But knowledge occurs in time, and we don't obtain all of our knowledge at once. Some is more difficult, so we need more time to deduce it, and some is trivial so can be had immediately. Without this temporal or related information of agent's knowledge, there is no way we can distinguish known propositions with respect to the difficulty of their deductions. Therefore we have to assume that agents possess all their knowledge at the same time, and accordingly know too much. Following this line of analysis, it is expected that the alternative epistemic logics proposed can validate inference rules of the following form:

(2) $$\frac{\vdash \phi \to \psi}{\vdash K^i\phi \to K^j\psi}$$ (Timed Logical Consequence Closure),

where i, j are natural numbers indicating the time instances when the agent obtains his knowledge of ϕ and ψ. The rule tells us that if the agent knows the premise of a logical implication at some time, then he knows the conclusion at a later time. The most important feature that these proposed logics should meet is that the time difference, $j - i$, should reflect in some way the computational difficulty of the implication.

With a rule of such kind we can determine what is known by the agent based on the temporal limitation of reasoning we set for the agent. In particular, we can argue what is the natural temporal elapse for a normal agent under normal circumstances to deduce his knowledge and then determine what logical consequences will naturally be known by agents, given their premise knowledge. In our previous discussions of philosophical treatment of the logical omniscience problem, we also see that Hintikka envisioned an objective definition of "surface tautology," with some kind of obviousness, can solve the logical omniscience problem. These tautologies are those that agents can certainly derive in limited time within normal circumstances. We have argued that Hintikka's own suggestion of checking the quantificational depth of formulas can't do the job. Here, however, our suggestion can fill the need. Once the time when the agent gets to know a formula due to his logical reasoning is explicitly expressed, this piece of information can be used to distinguish the surface tautologies from not. Thus in logics with rules of such kind, agents are not portrayed as logically omniscient. They do not know all the logical consequences of their knowledge within a reasonable time; however, they are rational agents capable of performing logical reasoning. They can continue to make efforts to know all the logical consequences of their knowledge, if temporal restrictions are not made.

The goal of this chapter is to present logics with the features we have just described. Logics introduced in this chapter will be modified from MEL, both semantically and syntactically. We call these logics tMEL, timed Modal Epistemic Logic, to indicate that the moment of time when knowledge is known is made explicit. In order to achieve our goal, for each MEL logic, there will be an associated axiomatic-like deductive machinery which is supposed to be employed by an agent to increase his knowledge. When an agent will know a formula depends on how long it will take for the agent to deduce the formula by this machinery, and agents with different initial logical knowledge, called logical bases, functioning as axioms in axiom systems have different logical strengths and will deduce different amounts of formulas. The framework of tMEL respects this diversity. Agents with different initial logical knowledge will be described by different logics and hence for each MEL logic, there is actually a collection of corresponding tMEL logics to be introduced.

Among all tMEL logics corresponding to the same MEL logic, there are always some which delineate agents with richer initial logical knowledge and have a close relationship with the MEL logic — theorems of MEL logic are exactly the theorems of these tMEL logics without considering the temporal components. This formal connection between MEL and tMEL is called the *realization theorem*. It shows that these MEL logics do have dynamic aspects themselves, but these aspects are only revealed in their tMEL counterparts. The implication of this connection theorem can be interpreted as follows. Knowledge acquisition happens in time, so a successful logic of knowledge must feature a temporal dimension of knowledge, implicitly or explicitly. Here the connection theorem justifies that MEL is truly a logic of the content of knowledge with a hidden temporal structure realized in tMEL. It rebuts the view that the problem of logical omniscience is inherent in possible world semantics, as many researchers suggest. The incapability of MEL, however, is the result of missing explicit temporal components from which model designers can determine what is known and what is not known by agents under various temporal restrictions, and then the success of applications of MEL will depend on how much temporal consideration should be taken into account in applications.

The Methodology and a Historical Note

The epistemic model we are going to present here is augmented from possible world semantics by adding a syntactical device to each world. Methodologically, this is similar to the awareness approach but with some necessary refinements. The syntactical device introduced here, also called awareness function by us, is not simply a sieve to distinguish knowledge of different types, explicit from implicit, but is structured to record the time when the agent is aware of formulas through deductive reasoning. In Fagin and Halpern (1988), it is once suggested the possibility of utilizing awareness functions such that time can be added. Our approach can be regarded as a direct response to this suggestion.

The study in this chapter, however, originates from another tradition. Justification Logic begins with *Logic of Proofs* LP (1995; 2001), introduced as an explicit proof counterpart of modal logic S4, which has a provability reading suggested by Gödel (1938). New formula constructors called *proof terms* or *proof polynomials* t, and new atoms of formulas $t{:}\phi$ meaning t is a proof, evidence, or justification of ϕ are introduced. The full definition of the language and proof system of LP can be found in the next chapter. Today Justification Logic is a well-developed area of research. Many technical results and extensions have been studied.[2] Syntactically, tS4, the tMEL counterpart of S4 that our following discussions are focusing on, with a suitable logical base, can be viewed as a numerical version of LP; and its axiom system is first introduced in Wang (2009b, 2011b), which constitutes the major part of the next chapter, for the study of the relationship between axiomatic proofs of S4 and LP. In Wang (2009b, 2011b), the axiom system tS4 is named $S4^{\Delta}$ and we will keep the name in the next chapter to detach its epistemic content and focusing on its syntactic properties. A constructive proof of the *realization theorem* between $S4^{\Delta}$ and S4 will be provided in the next chapter.

As we've mentioned in the introduction, that both Justification Logic and tMEL should pass the logically omniscient test introduced in Artemov and Kuznets (2006, 2009), and Justification Logic can also be regarded as an epistemic logical framework modeling rational agents and without the assumption of logical omniscience. With a suitable choice of the limitation of justificatory complexity, we can determine a natural class of logical consequences that a normal agent in normal circumstances can produce. In the next chapter, an efficient translation between $S4^{\Delta}$ and LP will be given. Interpreted epistemically, this result conforms to our intuition about the relation between justification and time. Justifications need time to produce, and we gain knowledge through time by giving justifications. Thus the study of tS4 and LP, or more broadly, the study between tMEL and Justification

[2] Cf. Artemov and Nogina (2005), Artemov (2006), Artemov (2008), Mkrtychev (1997), Fitting (2005a), Krupski (2006), and Brezhnev and Kuznets (2006), for instance.

Logic can complement each other. While Justification Logic has more sophisticated machinery representing justification by which the reasoning history of knowledge can be traced, reasoning with the temporal feature of knowledge is more intuitive, and working with natural numbers and their linearity is easier than directly dealing with justification terms.

The first possible world semantics with intended epistemic interpretation for Justification Logic is given by Fitting in (2005a), which our semantics of tMEL is mainly based on, but some major modifications will be made to have the semantics fit our intuition concerning knowledge in the passage of time. In this chapter, the focus will be on tS4, but the method can be uniformly extended to the study of other MEL and tMEL logics. At the end of this chapter we will discuss the semantic conditions for these logics, which include logics with the 5 Axiom, the so-called Negative Introspection Axiom.[3]

4.2 Logics of tS4

4.2.1 Possible World Semantics of S4

To begin, we review MEL and possible world semantics, which will form the basis of tMEL semantics. The language of MEL is built up from a set of primitive propositions, Boolean connectives $\sim$ and $\rightarrow$ ($\wedge$ and $\vee$ are defined in terms of the other two), and a modal operator K, together with parentheses for delimitation. In this chapter, we only consider the case of one agent and discuss propositional knowledge. *Knowledge* or *epistemic* will be used broadly to cover some cases in which *belief* or *doxatic* might be a better term. Well-formed formulas are defined as usual; in particular, if ϕ is a formula, so is $K\phi$, which means the agent knows ϕ, or ϕ is known. If no confusion will be made, the outmost parentheses are usually omitted. A *structure* or *a model* for MEL is a tuple $\langle W, R, \mathcal{V}\rangle$, where W is a set of worlds or epistemic alternatives, R is a binary relation defined on W, and $\mathcal{V}$ is a function assigning possible worlds to primitive propositions. The *satisfaction relation* in a structure M is recursively defined as follow:

1. $(M, w) \Vdash p$, where p is a primitive proposition iff $w \in \mathcal{V}(p)$,
2. $(M, w) \Vdash \sim\phi$ iff $(M, w) \nVdash \phi$,
3. $(M, w) \Vdash \phi \rightarrow \psi$ iff $(M, w) \nVdash \phi$ or $(M, w) \Vdash \psi$,
4. $(M, w) \Vdash K\phi$ iff $(M, w') \Vdash \phi$ for all $w' \in W$ with wRw'.

[3]The Justification Logic counterpart of Axiom 5 can be found in Pacuit (2005) and Rubtsova (2006).

We call a formula *valid in a structure* if it is satisfied in every world of the structure. Formulas which are valid in all structures compose the smallest MEL logic, *K*. For a more precise term, this is a logic of belief, not a logic of knowledge. Within this framework, to model agents under additional epistemic considerations, a subclass of structures are concerned. For example, in this chapter we study the knowledge of an agent with *positive introspection*, which is characterized by the *reflexive* and *transitive* structures. Formulas valid in these structures form *S4*, the logic of knowledge proposed by Hintikka. It is not difficult to see that in this semantics if an implication $\phi \rightarrow \psi$ is valid in all structures, so is $K\phi \rightarrow K\psi$. The agent modeled in this semantics knows all the logical consequences of his knowledge.

4.2.2 Awareness by Deduction

The language of tS4, as well as that of the framework of tMEL in general, is similar to the language of MEL, except that natural numbers are also formula constructors and if ϕ is a tMEL formula, $K^i\phi$, not simply $K\phi$, is a tMEL formula. The natural numbers are used to model the passage of time. We only consider a simple structure of time: discrete, linear, and with an initial point. The intended meaning of $K^i\phi$ is that the agent knows ϕ at time i, or ϕ is known at i. Notice that the natural numbers are only assigned to knowledge statements. Other formulas are considered temporal invariances. In particular, the facts of the world captured by the primitive propositions are supposed to be unchanged in the course of the agent's reasoning.

An *awareness function* is a partial function that associates tMEL formulas with natural numbers. Given a formula ϕ and a natural number i, $\alpha(\phi)=i$ means that the agent is aware of ϕ at time i and no earlier than i, i.e., the first time the agent is aware that ϕ is i. It is implicitly presupposed that the agent's awareness is monotone, that is, if the agent is aware of a formula at i, he is also aware of the formula at $i+1$. The choice of α as a partial, not total, function is to manifest that it is not necessary that the agent be aware of all formulas.

Awareness functions are employed to represent the agent's deduction history by recording the time when a formula is derived. Each deduction history has to start from some base formulas which are not derived from any others. The agent might have them inherently, or hear them from someone else. Given a tuple $\mathcal{A} = \langle \mathbf{A}, f \rangle$, where $\mathbf{A}$ is a set of formulas and f is a total function assigning formulas in $\mathbf{A}$ to a number, we say that an awareness function α is *based on* $\mathcal{A}$, or $\mathcal{A}$ is a *base* of α, if it satisfies the following:

0. If $A \in \mathbf{A}$, then $\alpha(A) \leq f(A)$. (*Initial Condition*)

We call $\mathbf{A}$ the *base set* of $\mathcal{A}$, and f the *base function* of $\mathcal{A}$. Sometimes we will simply write $\phi \in \mathcal{A}$ to mean $\phi \in \mathbf{A}$. We call α an *$\mathcal{A}$-awareness function*, or just write $\alpha_{\mathcal{A}}$, to indicate α is based on $\mathcal{A}$.

The most basic rule that we assume our agent can perform is *Modus Ponens* and for each step, he will use a unit of time to do it. We model these as ($\alpha(\phi)\downarrow$ means $\alpha(\phi)$ is defined):

1. If $\alpha(\phi \to \psi)\downarrow$ and $\alpha(\phi)\downarrow$, then

$\alpha(\psi) \leq \max(\alpha(\phi \to \psi), \alpha(\phi)) + 1.$ (*Deduction by Modus Ponens*)

The reason we use *less-than-or-equal-to* ($\leq$) and not simply *equal-to* ($=$) in the main clauses of the rules is that the agent might be aware of, say, ψ earlier in this case, since he has other resources and may derive it from other formulas.

Another general rule that we assume our agent can manipulate is that for base formulas, he is able to be aware that he knows them. Suppose, for example, someone at time i told the agent ϕ, and from this the agent is able to deduce "he knows ϕ at i." This deduction rule is formulated as:

2. If $A \in \mathbf{A}$ and $f(A) \leq i$, then

$\alpha(K^i A) \leq i + 1.$ (*Deduction by $\mathcal{A}$-Epistemization*)

An $\mathcal{A}$-awareness function that satisfies the above conditions is called *normal*.

Finally, for an S4 (or tS4) agent with positive introspection, i.e., knowing what he knows, we assume he is able to be aware that he knows ϕ for any formula ϕ that he is aware of. This is modeled as follows:

3. For any ϕ, if $\alpha(\phi) \leq i$, then

$\alpha(K^i\phi) \leq i + 1.$ (*Inner Positive Introspection*)

The word *inner* implies there will be an *outer* rule. We call the rule that the agent can perform here *inner* since it is the introspection about the awareness of a formula. The *outer* rule, which will be defined after we introduce the semantics, is the introspection about the satisfaction of the knowledge of a formula.

It is also apparent that the *Inner Positive Introspection* condition is a general form of the condition *Deduction by $\mathcal{A}$-Epistemization.* We separate them here to demonstrate epistemically that the latter is more basic than the former (only formulas assumed in the base can be epistemized). The advantage of this separation will become clearer when we consider logics without Positive Introspection.

Given these conditions, we can build an awareness function which is purely affected by the formulas in the base. To some extent, this function is a complete characterization of our agent's reasoning ability. For two awareness functions α and β, we write $\beta \leq \alpha$, if $\beta(\phi) \leq \alpha(\phi)$ for any formula ϕ such that $\alpha(\phi)\downarrow$.

Lemma 4.2.1. *Given an awareness base $\mathcal{A}$, there exists a unique normal (S4, or none) $\mathcal{A}$-awareness function $\alpha^*_{\mathcal{A}}$, called* critical*, such that for any $\mathcal{A}$-awareness function α, $\alpha \leq \alpha^*_{\mathcal{A}}$.*

Informally speaking, the critical awareness function makes an agent aware of a formula as late as possible. The agent might be aware of the formula earlier if additional information is possessed.

4.2.3 Semantics

We first consider the semantics for tMEL in general and then discuss the semantics for tS4 in particular. A *structure for tMEL* is a tuple $M = \langle W, R, \mathfrak{A}, \mathcal{V}\rangle$ where $\langle W, R, \mathcal{V}\rangle$ is a MEL structure and $\mathfrak{A}$ is a collection of awareness functions α_w indexed by the worlds $w \in W$. That is, for every world $w \in W$, there is one and only one $\alpha_w \in \mathfrak{A}$. The *satisfaction relation* of this semantics is defined as follows:

1. $(M, w) \Vdash p$, where p is a primitive proposition, iff $w \in \mathcal{V}(p)$;
2. $(M, w) \Vdash {\sim}\phi$ iff $(M, w) \nVdash \phi$;
3. $(M, w) \Vdash \phi \to \psi$ iff $(M, w) \nVdash \phi$ or $(M, w) \Vdash \psi$;
4. $(M, w) \Vdash K^i\phi$ iff $(M, w') \Vdash \phi$ for all $w' \in W$ with wRw', and $\alpha_w(\phi) \leq i$.

We can easily recognize that only the last clause is altered from the standard MEL satisfication relation. It states that at time i, the agent knows ϕ in a world w if and only if the formula is true at all worlds accessible from w, and that the first time he is, by deduction or other means, aware of the formula is before or at i. Of course, setting the rule in this way, we presuppose that the agent won't forget what he knows. One reason for this setting is to simplify the argument; the other is that we consider the scenario in which the agent is goal-directed, directing all his efforts in logical reasoning, and hence in the course of the reasoning, it is assumed that he won't forget what he knows. Nevertheless, this presupposition is not essential to the construction of the framework; systems without the presupposition can be developed by adjusting the condition set here.

Similar to MEL, agents are classified by subclasses of tMEL-structures. But subclasses now are also determined by the collections of awareness functions in the structures. We attribute properties to collections of awareness functions by their elements. For example, if a collection contains only normal awareness functions, we say the collection is *normal*, and if the collections contains only $\mathcal{A}$-awareness functions, we say the collection is an $\mathcal{A}$-*collection*, or is *based on* $\mathcal{A}$. For a tMEL-structure $M = \langle W, R, \mathfrak{A}, \mathcal{V} \rangle$, we say $\mathfrak{A}$ is *monotonic* if for any wRw', $\alpha_{w'} \leq \alpha_w$, that is, accessible worlds have the agent aware of formulas at an earlier time.

Definition 4.2.2. *Given a base $\mathcal{A}$, we call a tMEL-structure $\langle W, R, \mathfrak{A}, \mathcal{V} \rangle$ a* tS4($\mathcal{A}$)-structure *if R is transitive and reflexive, and $\mathfrak{A}$ is normal, inner positive introspective, monotonic, and based on $\mathcal{A}$.*

The idea behind this semantics should be clear and intuitive. In each world, we have the valuation function to tell us the truth or falsity of the outside world, and have the awareness function to tell us the dynamic mental behavior of the agent. In this way, the agent's varied epistemic abilities are also manifested by diverse relations between awareness functions in structures. For example, monotonicity says that the agent can only conceive of a world in which he himself might be aware of some formula at an earlier time because he might have some additional information in the conceived world, but he won't conceive of a world in which he himself is aware of fewer formulas since he, as an S4 agent, has strong evidence as to what he is aware of in this world.

We say a formula is *valid* in a tMEL-structure if the formula is satisfied at all worlds in the structure, and a formula is *tS4($\mathcal{A}$)-valid* if it is valid in all tS4($\mathcal{A}$)-structures. We denote this as $\vDash_{tS4(\mathcal{A})} \phi$, given that $\mathcal{A}$ is a base. Then the *logic of tS4($\mathcal{A}$)* is the set of tS4($\mathcal{A}$)-valid formulas.

Here are some properties of our model of knowledge and time, for an S4 agent. They are valid in all tS4-structures.

1. Classical tautologies;
2. $K^i(\phi \to \psi) \to (K^j\phi \to K^k\psi)$ for $i, j < k$;
3. $K^iA \to K^j(K^iA)$ $i < j$ if $A \in \mathcal{A}$;
4. $K^i\phi \to K^j\phi$ $i < j$;
5. $K^i\phi \to \phi$;
6. $K^i\phi \to K^j(K^i\phi)$ $i < j$.

The validity of almost of all these formulas directly follows the definitions of awareness function, satisfication relation, and structure. We prove the last, which needs some care. Suppose $K^i\phi$ is true at some world w and wRw'. By the definition of satisfication relation, $\alpha_w(\phi) \leq i$ and ϕ is true in w'. It then follows that $K^i\phi$ is true at every w' with wRw', since R is transitive and $\alpha_{w'}(\phi) \leq \alpha_w(\phi) \leq i$ ($\mathfrak{A}$ is monotonic). Furthermore, since α_w is inner positive introspective, $\alpha_w(K^i\phi) \leq i+1$. It concludes that $K^j(K^i\phi)$ is true in w when $i < j$. Note that in the proof we need both the conditions that the collection of awareness functions is monotonic and inner positive introspective.

For a tMEL-structure $\langle W, R, \mathfrak{A}, \mathcal{V}\rangle$, we say $\mathfrak{A}$ is *outer positive introspective* if all the awareness functions α_w in $\mathfrak{A}$ satisfy the following condition: if $(M, w) \Vdash K^i\phi$, then $\alpha_w(K^i\phi) \leq i+1$. It is an easy exercise to show that if $\mathfrak{A}$ is inner positive introspective, then it is outer positive introspective. The reason we introduce this outer rule is to make comparisons with the situation that occurs when we deal with the negative introspection condition. In that case, the outer rule implies the inner rule. We give a definition for the package of conditions on the collection of awareness functions that are related to positive introspection.

Definition 4.2.3. *For a tMEL-structure $\langle W, R, \mathfrak{A}, \mathcal{V}\rangle$, we say $\mathfrak{A}$ is* positive regular *if it is normal, monotonic, and both inner and outer positive introspective.*

Then a tMEL structure is a tS4($\mathcal{A}$)-structure if and only if R is reflexive and transitive, and $\mathfrak{A}$ is positive regular.

4.2.4 Logical Bases

We now turn our focus to the bases of awareness functions. The formulas in the base of an awareness function are the formulas of which we suppose the agent is intrinsically aware. We have not yet placed any restriction on the base in our definition. Therefore, bases can be composed of empirical facts, which are captured by primitive propositions, or of inconsistencies. But for now we are interested in bases which contains only logical truths, i.e., valid formulas. Later, tS4($\mathcal{A}$) logics with bases $\mathcal{A}$ of logical truths will be axiomatized.

To decide what could be counted as a logical base is not a straightforward task. Certainly, all formulas valid in all tS4 structures are logical truths, and with this definition we already have some interesting logical bases. For example, we can study the logic of an agent aware of all classical or intuitionistic tautologies; these logics are worth further study. However, we would like our definition to be more comprehensive. Consider the following case. Suppose the propositional tautology $\phi{=}A{\to}(B{\to}A)$ is the only element in a base $\mathcal{A}$, and then it can be shown that $\psi = K^i A \to K^j (B \to A)$ is a tS4($\mathcal{A}$)-valid formula for some $i < j$, but not valid in all tS4 structures. However, there seems no reason to exclude a base containing only formulas ϕ and ψ as a logical base. We will give a constructive definition of logical bases hinted at by this example.

Here is some terminology, most of which is standard. Given bases $\mathcal{A} = \langle \mathbf{A}, f \rangle$ and $\mathcal{B} = \langle \mathbf{B}, g \rangle$, $\mathcal{B} \subseteq \mathcal{A}$ means $\mathbf{B} \subseteq \mathbf{A}$ and $f(B) \leq g(B)$ for all $B \in \mathbf{B}$. We call a set of bases $\{\mathcal{A}_i (= \langle \mathbf{A}_i, f_i \rangle)\}_{i \in \mathbb{N}}$ an *ascending chain* if $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \ldots$, and we say a base $\mathcal{A}$ is the *limit* of the ascending chain if $\mathcal{A} = \bigcup \mathcal{A}_i$, i.e., $\mathbf{A} = \bigcup \mathbf{A}_i$ and $f(A) = \min\{f_i(A) : f_i(A)\downarrow\}$.

We also say a base $\mathcal{A}$ is *(tS4-)sound over* a base $\mathcal{B}$ if for every $\phi \in \mathcal{A}$, $\models_{tS4(\mathcal{B})} \phi$, and we say $\mathcal{A}$ is *sound and completely over* $\mathcal{B}$ if $\mathcal{A}$ is sound over $\mathcal{B}$ and for every tS4($\mathcal{B}$)-valid formula ϕ, $\phi \in \mathcal{A}$.

Definition 4.2.4. *To say that a base $\mathcal{A}$ is* tS4 logical *means at least one of following is true: (1) $\mathcal{A}$ is empty, (2) $\mathcal{A}$ is sound over a tS4 logical base, or (3) $\mathcal{A}$ is the limit of an ascending tS4 logical bases $\{\mathcal{A}_i\}_{i \in \mathbb{N}}$, where A_{i+1} is sound over A_i for each $i \in \mathbb{N}$.*

If $\mathcal{B} \subseteq \mathcal{A}$, every tS4($\mathcal{B}$)-valid formula is a tS4($\mathcal{A}$)-valid formula. So it is not difficult to see that if $\mathcal{A} = \langle \mathbf{A}, f \rangle$ is a logical base, then for every $\phi \in \mathbf{A}$, $\models_{tS4(\mathcal{A})} \phi$ and hence $\models_{tS4(\mathcal{A})} K^i \phi$ for $f(\phi) \leq i$. Also notice that the definition of logical bases is logically dependent. For different tMEL logics, there will be different logical bases to be concerned. We have the following lemma:

Lemma 4.2.5. *Given a tS4 logical base $\mathcal{A}$, and a tMEL formula ϕ, if $\alpha^*_{\mathcal{A}}(\phi)$ is defined, then ϕ is tS4($\mathcal{A}$)-valid.*

Logical bases also have a finitude feature. Call a logical base finite if its base set is finite.

Lemma 4.2.6. *Given a tS4 logical base $\mathcal{A}$ and a tMEL formula ϕ, $\models_{tS4(\mathcal{A})} \phi$ if and only if there is a finite tS4 logical base $\mathcal{B} \subseteq \mathcal{A}$ such that $\models_{tS4(\mathcal{B})} \phi$.*

This lemma can be proved semantically by first proving the compactness theorem. To save space, we leave it as a corollary of the completeness theorem, which we will show later when axiom systems are introduced.

4.3 More on Logical Bases

One advantage of the framework that we introduced above is its flexibility in modeling agents with different initial logical knowledge (logical bases).

Given a tS4 logical base $\mathcal{A}$, we have the logic tS4($\mathcal{A}$) particularly describes the logical and temporal structure of the knowledge possessed by an agent with the logical strength determined by the logical base. In this section, we discuss several natural conditions on the logical bases, and some of these conditions, as we will show, turn out to determine the same class of logical bases.

The smallest logical base is the empty base, composed of the empty set and the empty function. It is clear and supported by the logic that an agent with empty base does not have any assured knowledge. No tS4($\emptyset$)-valid formula is of the form $K^i\phi$. One thing should be clarified: given a logical base $\mathcal{A}$, the logic tS4($\mathcal{A}$) is not exclusively about the agent with base $\mathcal{A}$. Instead, it is a logic of an agent who has *at least* $\mathcal{A}$ as his base. Hence a formula that is tS4($\emptyset$)-valid is also tS4($\mathcal{A}$)-valid for every $\mathcal{A}$.

At the other extreme, there are logical bases in which every logical truth has been included. We call a tS4 logical base $\mathcal{A}$ *comprehensive* if for every tS4($\mathcal{A}$)-valid formula ϕ, $\phi \in \mathcal{A}$. Given an ascending chain of tS4 logical bases $\{\mathcal{A}_i\}_{i\in\mathbb{N}}$ where $\mathcal{A}_0$ is empty, and for every $i \in \mathbb{N}$, $\mathcal{A}_{i+1}$ is sound and completely over $\mathcal{A}_i$, then the limit of the ascending chain is comprehensive. This is a direct result following the finitude of logical bases. Since if $\models_{tS4(\mathcal{A})} \phi$, then, by Lemma 4.2.6, $\models_{tS4(\mathcal{A}_i)} \phi$ for some i, so $\phi \in \mathcal{A}$. Note that there is more than one comprehensive base. In the above construction, if the base functions of the logical bases in the chain are changed, we shall get different comprehensive logical bases. Among all these comprehensive bases, there is a maximal one. Call a base *principal* if its base function is the constant function 0. Let $\mathcal{A}$ be a tS4 logical base which is comprehensive and principal. Then it is not difficult to see that for any comprehensive tS4 logical base $\mathcal{B}$, $\mathcal{B} \subseteq \mathcal{A}$. We can also have the following result immediately: if $\models_{tS4(\mathcal{A})} \phi$, then $\models_{tS4(\mathcal{A})} K^i\phi$ for any $i \in \mathbb{N}$.

Agents with comprehensive logical bases are unrealistic. They know too much from the beginning. So for realistic logical bases, some moderate conditions should be satisfied. Generally speaking, we would like the logical base to be smaller in size but without limiting the reasoning ability of the agent. One of these conditions is to demand that agents with logical bases of this kind be logically indistinguishable from agents with comprehensive bases. That is, given a tS4 logical base $\mathcal{A}$, the following is satisfied:

(i) There is a comprehensive base $\mathcal{B}$ such that $\models_{tS4(\mathcal{A})} \phi$ iff $\models_{tS4(\mathcal{B})} \phi$.

The other consideration comes from the concern of the awareness function. Since, in the semantics, the awareness function simulates the agent's deductive ability, we would like a logical base to be rich enough such that an agent with the base is able to be aware of every valid formula. This condition can be formulated in two ways:

(ii) If $\models_{tS4(\mathcal{A})} \phi$, then $\alpha^*_{\mathcal{A}}(\phi)\downarrow$, or

(ii') If $\models_{tS4(\mathcal{A})} \phi$, then $\{\alpha(\phi) \mid \alpha$ is a tS4 $\mathcal{A}$-awareness function and $\alpha(\phi)\downarrow\}$ is a finite set.

Condition (ii) is the converse of Lemma 4.2.5. It might be helpful to think of these conditions as follows: Lemma 4.2.5 says that if a base is logical, then it is sound, and our conditions here state that we would like the base to be complete, too. Of course, this way of speaking is nonstandard, since the agent's inference system is part of the semantics.

Finally, we might want our agent to have a logical base rich enough such that all the logical truths are known to him in every world in every structure. That is:

(iii) If $\models_{tS4(\mathcal{A})} \phi$, then $\models_{tS4(\mathcal{A})} K^i\phi$, for some $i \in \mathbb{N}$.

Now these conditions come from different considerations; however, the interesting thing is that they determine the same class of tS4 logical bases. We will call a base *full* if it satisfies one of these conditions.

Theorem 4.3.1. *The conditions (i), (ii), (ii'), and (iii) categorize the same class of logical bases.*

Proof. The equivalences between conditions (ii), (ii'), and (iii) are straightforward so we will prove the equivalence between conditions (i) and (ii). We first prove the direction from (i) to (ii). Given a tS4 logical base $\mathcal{A}$, suppose that $\models_{tS4(\mathcal{A})}\phi$ and that there is a comprehensive logical base $\mathcal{B}$ such that $\models_{tS4(\mathcal{A})}\psi$ iff $\models_{tS4(\mathcal{B})}\psi$ for every ψ. Then $\models_{tS4(\mathcal{B})}\phi$. Since $\mathcal{B}$ is comprehensive, so $\phi \in \mathcal{B}$ and hence $\models_{tS4(\mathcal{B})}K^i\phi$ for $f(\phi) \leq i$ where f is the base function of $\mathcal{B}$. Now, following the assumption, $\models_{tS4(\mathcal{A})}K^i\phi$ for some i and hence $\alpha^*_{\mathcal{A}}(\phi)$ is defined. This completes the proof in one direction. For the other direction, suppose that for every tS4($\mathcal{A}$)-valid formula ϕ, $\alpha^*_{\mathcal{A}}(\phi)$ is defined. We define $\mathcal{B} = \langle \mathbf{B}, g\rangle$ with $\mathbf{B}=\{\phi \mid \models_{tS4(\mathcal{A})} \phi\}$ and $g(\phi)=\alpha^*_{\mathcal{A}}(\phi)$. Since for any $\phi \in \mathbf{B}$ and any tS4 $\mathcal{A}$-awareness function α, $\alpha(\phi) \leq g(\phi)(= \alpha^*_{\mathcal{A}}(\phi))$, α is a tS4 $\mathcal{B}$-awareness function. Hence every tS4($\mathcal{A}$)-structure is a tS4($\mathcal{B}$)-structure and every tS4($\mathcal{B}$)-valid formula is an tS4($\mathcal{A}$)-valid formula. By the definition of $\mathcal{B}$, every tS4($\mathcal{A}$)-valid formula is in $\mathcal{B}$. $\mathcal{B}$ is comprehensive. ⊣

Logics with full bases will have the desirable property for epistemic logic as we discussed in the introduction. The rule of Timed Logical Consequence Closure is sound in these logics. Given a tS4 full logical base $\mathcal{A}$ and suppose $\models_{tS4(\mathcal{A})}\phi\to\psi$, then $\models_{tS4(\mathcal{A})}K^k(\phi\to\psi)$ for some k, since $\mathcal{A}$ is full. From the construction of the semantics, especially the rule of Deduction by Modus

Ponens of awareness functions, it is not difficult to see that if the agent knows ϕ at some time i, then he is able to know ψ at a later time j.

Lemma 4.3.2. *Suppose $\mathcal{A}$ is a full logical base, then the following rule holds:*

$$\frac{\vDash_{tS4(\mathcal{A})} \phi \rightarrow \psi}{\vDash_{tS4(\mathcal{A})} K^i\phi \rightarrow K^j\psi} \text{ for some } j > i.$$

In the next section, after the axiom systems are introduced, we will see a concrete example of a full logical base.

4.4 Axiomatization

Now we consider the syntactical counterparts of the semantic structures we given above. It is actually infinitely many logics with their semantics that have been introduced. Nonetheless, they can be axiomatized in a uniform way. Given a tS4 logical base $\mathcal{A}=\langle \mathbf{A}, f\rangle$, the corresponding axiom system of tS4($\mathcal{A}$) is the following:

Definition 4.4.1. *tS4($\mathcal{A}$) Axiom Systems*

Axioms	
A0 Classical propositional axiom schemes	
A1 $K^i(\phi \rightarrow \psi) \rightarrow (K^j\phi \rightarrow K^k\psi) \quad i, j < k$	*(Deduction by Modus Ponens)*
A2 $K^iA \rightarrow K^j(K^iA) \quad i < j$ *if* $A \in \mathbf{A}$ *and* $f(A) \leq i$	*(Deduction by $\mathcal{A}$-Epistemization)*
A3 $K^i\phi \rightarrow K^j\phi \quad i < j$	*(Monotonicity)*
A4 $K^i\phi \rightarrow K^j(K^i\phi) \quad i < j$	*(Positive Introspection)*
A5 $K^i\phi \rightarrow \phi$	*(Truth Axiom)*

Inference Rules	
R1 if $\vdash \phi \rightarrow \psi$ *and* $\vdash \phi$, *then* $\vdash \psi$	*(Modus Ponens)*
R2 if $A \in \mathbf{A}$ *and* $f(A) \leq i$, *then* $\vdash K^iA$	*($\mathcal{A}$-Epistemization)*

The validity of these axioms is easily established, and the soundness of the rule of Modus Ponens is evident. Since we only consider logical bases, the rule of $\mathcal{A}$-Epistemization is automatically justified. So the soundness naturally follows. Let $\vdash_{tS4(\mathcal{A})} \phi$ denote that ϕ is a theorem of the axiom system tS4($\mathcal{A}$).

Theorem 4.4.2. *Given a tS4 logical base $\mathcal{A}$, $\vdash_{tS4(\mathcal{A})} \phi$ if and only if $\vDash_{tS4(\mathcal{A})} \phi$.*

Proof. We prove the completeness part of this theorem. We will construct a tS4($\mathcal{A}$)-structure composed of maximal $\mathcal{A}$-consistent sets. A set S of tMEL formulas is said to be *$\mathcal{A}$-consistent* if there is no finite subset $\{F_1, \ldots, F_n\}$ of S such that $\neg(F_1 \wedge \ldots \wedge F_n)$ is a tS4($\mathcal{A}$) theorem. The construction of a maximal such set is by the standard Lindenbaum construction. Let W

be the set of all maximal $\mathcal{A}$-consistent sets and for any Γ, $\Gamma' \in W$, we define $\Gamma R \Gamma'$ if and only if $\Gamma^\sharp \subseteq \Gamma'$, where $\Gamma^\sharp = \{F \mid K^i F \in \Gamma\}$, and define functions α_Γ and $\mathcal{V}$ by setting $\alpha_\Gamma(F)=\min\{i \mid K^i F \in \Gamma\}$ and $\mathcal{V}(P)=\{\Gamma \mid P \in \Gamma\}$. We claim this $M=\langle W, R, \mathfrak{A}, \mathcal{V}\rangle$ with every $\alpha_\Gamma \in \mathfrak{A}$ is a tS4($\mathcal{A}$)-structure. The transitivity and reflexivity of R is implied by Positive Introspection and Truth Axiom. The $\mathcal{A}$-Epistemization rule implies that α_Γ is an $\mathcal{A}$-awareness function. It is not difficult to check that α_Γ also satisfies other conditions by applying the corresponding axioms of these conditions. Finally, the collection of these awareness functions also satisfies the monotonicity condition. Suppose $\alpha_\Gamma(F) = i$, $K^i F \in \Gamma$. Since $\vdash_{tS4(\mathcal{A})} K^i F \to K^j(K^i F)$, $K^j(K^i F) \in \Gamma$, so $K^i F \in \Gamma'$ for any $\Gamma^\sharp \subseteq \Gamma'$. $\alpha_{\Gamma'}(F) \leq i$.

We now prove Truth Lemma: for every Γ, $F \in \Gamma$ if and only if $(M, \Gamma) \Vdash F$. The proof is by induction and most cases are trivial. We prove the modal case. If $(M, \Gamma) \Vdash K^i F$, then $\alpha_\Gamma(F) \leq i$, so $K^i F \in \Gamma$. For the other direction, if $K^i F \in \Gamma$, $\alpha_\Gamma(F) \leq i$ and for any $\Gamma^\sharp \subseteq \Gamma'$, $F \in \Gamma'$, so by Induction Hypothesis, $(M, \Gamma') \Vdash F$, and hence $(M, \Gamma) \Vdash K^i F$. This completes the proof of Truth Lemma. Now suppose ϕ is not provable in tS4($\mathcal{A}$), $\neg\phi$ is $\mathcal{A}$-consistent. $(M, \Gamma) \nVdash \phi$ with Γ a maximal $\mathcal{A}$-consistent set containing $\neg\phi$. ϕ is not tS4($\mathcal{A}$)-valid. ⊣

When the awareness base has some special property, the description of the systems can be simplified. For example, when $\mathcal{A}$ is empty, the A2 axiom and R2 rule are void. When $\mathcal{A}$ is comprehensive, the clause "$A \in \mathcal{A}$" can be replaced by "$\vdash A$," and when the base is principal, "and $f(A) \leq i$" can be removed.

It is not difficult to recognize that this axiom system is almost completely parallel to the standard S4 axiom system, except that the language is richer and for each axiom, there are additional conditions on the temporal components. Each axiom describes one reasoning ability that the agent in discussion processes. Exceptions are Monotonicity (not to be confused with the monotonicity of collections of awareness functions) and Truth Axiom, which are descriptive properties of knowledge. The rule of $\mathcal{A}$-Epistemization is probably the most atypical. However, if the logical base $\mathcal{A}$ is comprehensive, then the rule directly corresponds to the Necessitation Rule of S4.

Interestingly, this type of axiom system itself gives us a logical base to consider, that is, the logical base which contains all these axioms. We will show below that it is full.

Lemma 4.4.3. *Given a tS4 logical base $\mathcal{A}$, if every axiom instance belongs to $\mathcal{A}$, $\mathcal{A}$ is full.*

Proof. With the completeness and soundness results above, it is sufficient to prove that if ϕ is a theorem, then $K^i\phi$ for some i is also a theorem. We prove the statement by induction on the length of the proof of ϕ. Suppose ϕ is an axiom. Then by $\mathcal{A}$-Epistemization, $K^i\phi$ for $i \geq f(A)$ is a theorem. If ψ is derived from $\phi \to \psi$ and ϕ, then, by the Induction Hypothesis, both $K^i(\phi \to \psi)$ and $K^j\phi$ are theorems. Using axiom A1, we have a theorem $K^k\psi$ for $k > i, j$. If $K^i\phi$ is derived by $\mathcal{A}$-Epistemization, by applying axiom A2, $K^j(K^iF)$ for $j > i$ is a theorem. ⊣

Definition 4.4.4. *We say a tS4 logical base $\mathcal{A}$ is* axiomatically appropriate *if it contains all axiom instances of the schemes listed in the above system.*

When $\mathcal{A}$ is axiomatically appropriate, the clause "$A \in \mathcal{A}$" can be replaced by "A is an axiom" in the axiom system.

Notice how axioms A1 and A2 are the only axioms used in the proof of Lemma 4.4.3. This result meshes with our intuition. Suppose that the agent we are going to investigate is aware of these axioms, that is, the agent's logical strengths are just like ours, and A1 and A2 explain that the agent can reason using Modus Ponens and $\mathcal{A}$-Epistemization. Then, with whatever formula we can prove ($\vdash \phi$), the agent can prove as well. So the agent is aware of ϕ at some time and hence the agent knows it ($\vdash K^i\phi$, for some i).

These considerations explain why we list A2 separate from A4. When we consider logics without Positive Introspection, the A2 axiom plays a pivotal role in ensuring that the axiomatically appropriate bases of these logics are still full. For example, consider the axiom system tK, which consists of the tS4 axiom system without Positive Introspection and the Truth Axiom; tK is the tMEL counterpart of the minimal modal epistemic logic K. Lemma 4.4.3 will still hold if we take $\mathcal{A}$ to be a tK logical base.

Realization Theorem

Our semantic work for tS4 is a modified possible world semantics, and it is clear that the tS4 axiom systems displayed above is quite parallel to the standard axiom system S4. Now the question is, is there any formal relation between the logics tS4 and S4? As we have seen, tS4 is actually a collection of logics differentiated by their logical bases. Agents with different logical bases have different logical strengths. So it can be expected that not every tS4 logic has a direct formal connection with S4. It is shown in Wang (2009b) that the tS4 logic with a logic base which is principal and axiomatically appropriate can *realize* all S4 theorems.

Theorem 4.4.5 (The Realization Theorem)**.** *Let $\mathcal{A}$ be a logical base that is principal and axiomatically appropriate. A MEL formula ϕ is an S4 theorem* *if and only if there is a corresponding tMEL formula ψ which is a tS4($\mathcal{A}$)-theorem such that ϕ is the resulting formula from removing all the time labels in ψ.*

As we have mentioned, the study of tMEL is developed from the study of Justification Logic, and it is well-known that the realization theorem of Justification Logic have been established. In the next chapter, which is based on Wang (2009b), a new constructive realization procedure will be provided, from which both the tMEL and Justification Logic version of the realization theorem will be derived. A semantic proof of this theorem is a subject of future work. Experience suggests that the fullness of the logical base should be a sufficient condition for tS4 realization of S4 theorems.

In the context of this chapter, this theorem can be understood as the Temporalization Theorem, which gives us a new insight into modal epistemic logics. As we've argued, knowledge is accumulated over time, and MEL has its intuitive appealing and has been proved useful in cases, but is also considered as too idealized. Then the question is what the relation between MEL and the reasoning time is, and this theorem gives us an answer to this question. Although in MEL, we can't directly get the temporal information related to knowledge statements, but there is indeed temporal structure hidden in MEL and only realized in tMEL. Each MEL theorem only states the logical relation between the agent's known propositions, and when these known propositions are known is recorded in tMEL. The problem of MEL is therefore its lack of expressivity. When it is applied, since the temporal dimension is missing, all the known propositions have to be supposed to be known at the same time, and hence agents depicted by MEL are usually considered as knowing to much. But in tMEL, their rationalities which are displayed in MEL are retained and what is really known is temporal relevant.

4.5 More Logics

In this section, we discuss how to extend the framework we introduced above to other MEL logics and their tMEL counterparts. It is well known that in MEL there is a corresponding relation between axiom schemes and conditions on the binary relations in structures. For every axiom system with a special axiom scheme, its sound and complete semantic counterpart will be the subclass of structures in which the binary relation satisfies the corresponding condition of the axiom scheme. The situation is similar for tMEL, but with some subtleties. The axiom schemes under consideration are not the schemes in MEL, but its tMEL counterparts. For the convenience of comparison, we will call the the axiom of Positive Introspection in the tS4 axiom systems *t4 Axiom*, and Truth Axiom *tT Axiom*. Corresponding semantic conditions of tMEL axiom schemes will be on both binary relations and collections of awareness functions. The nice thing is that the conditions on the binary relations of the tMEL axioms are the same as those conditions of the MEL counterparts of these tMEL axioms. So the binary relation of axiom tT is reflexive, and that of t4 is transitive. The basic condition on the collection of awareness functions is normal. No additional condition is needed for tT. But for t4, we need the collection to be positive regular. Below we list the needed conditions for logics combining these axioms. Let $M = \langle W, R, \mathfrak{A}, \mathcal{V} \rangle$ be a tMEL structure (notation note: tKT4 is the logic tK with additional axioms tT and t4, and other logics are named similarly):

	R	$\mathfrak{A}$
tK	no condition	normal
tKT	reflexive	normal
$tK4$	transitive	positive regular
$tKT4$ (tS4)	transitive and reflexive	positive regular.

Now consider t5 Axiom: "$\neg K^i F \rightarrow K^j(\neg K^i F)$ for $i < j$." Given a structure $M = \langle W, R, \mathfrak{A}, \mathcal{V} \rangle$, we say $\mathfrak{A}$ is *inner negative introspective* if for every $\alpha_w \in \mathfrak{A}$, $\alpha_w(\neg K^i \phi) \leq i+1$, provided $\alpha_w(\phi) \not\leq i$, say $\mathfrak{A}$ is *anti-monotonic* if for any $w\mathcal{R}w'$, $\alpha_w \leq \alpha_{w'}$, and say $\mathfrak{A}$ is *outer negative introspective* if for every $\alpha_w \in \mathfrak{A}$, $\alpha_w(\neg K^i F) \leq i+1$, provided $(M, w) \Vdash \neg K^i F$; we then call $\mathfrak{A}$ *negative regular* if it is anti-monotonic and has both inner and outer negative introspection. For a tMEL logic with t5 axiom, its sound and complete semantics counterpart is the subclass of structures with *euclidean* binary relation and negative regular collection of awareness functions. It is easy to check that if $\mathfrak{A}$ is outer negative introspective, it is inner, and not the other way around.

Although theoretically we can come up with tMEL logics with t5 Axiom, from the epistemic point of view the axiom is rather dubious. The outer negative introspection rule says that the agent is aware of some formula in a world when some other formula is *satisfied* in the world. Unlike the inner rule in which the agent is aware of a formula because he is *aware* of another formula by his deduction and hence the rule can be considered as the agent reflects his reasoning process, the outer rule gives no hint as to what kind of reasoning process the agent is introspecting. There might be one such process (by which the agent learns that $\neg K^i F$ is satisfied in the world), but it is not in the model. Then a structure in which the collection of awareness functions is negative regular can also describe an agent who happens to be systematically aware of some formulas which are true at a world. Further work needs to be done to make t5 case.

4.6 Discussions

It probably isn't too much of an exaggeration to say that the logical omniscience problem is the most important threat to the enterprise of epistemic logic. From the beginning, philosophers have questioned the possibility of epistemic logic through this problem.[4] Later, epistemic logic finds its applications in many practical studies, but it is always argued that the knowledge modeled in epistemic logic is too idealized. The most successful applications of epistemic logic is in the study of distributed systems. One reason for this success is that the logical omniscience problem is not a problem in the application. The processes in systems don't produce their own knowledge. It is up to us, the model designers, to ascribe external knowledge to these processes for the study of the structures and behaviors of the systems. However, when we apply epistemic logic to intelligent agents, things are different. They are supposed to produce their own knowledge, and none of them can create as much knowledge as suggested by the standard modal epistemic logic. These seem to provide additional evidence that what matters is the computational efforts that the agent should make to produce knowledge, and that the temporal dimension of knowledge can help to reveal the information.

In this chapter, we introduced a general epistemic framework tMEL to deal with the problem of logical omniscience. tMEL is the explicit temporal counterpart of MEL. Its semantics is augmented from the standard possible world semantics such that each world is equipped with an awareness function to tell us how the agent deduces his knowledge over time, besides a valuation function assigning the primitive truths of the world. With the formalism of tMEL we can express some meaningful epistemic statements which can't be stated in MEL. For example, the monotonicity of knowledge can be expressed as $K^i\phi \rightarrow K^j\phi$ for $i < j$. Also tMEL sustain a great range of logics, each of which reasons about the epistemic aspects of agents with different reasoning strengths determined by their initial logical knowledge. These diverse logics, even disregarding their temporal structures, are worthy of their own study.

In the literature, Elgot-Drapkin and Perlis (1990) is an early attempt of introducing epistemic formalism that takes the agent's reasoning time into consideration. In the proposal, the first order formalism is taken and agent's object language is distinguished from the modeler's meta-language; it is in the latter that a binary knowledge predicate is introduced with an object language formula and the time when the formula is deduced as the arguments. Temporal Epistemic Logic is another formalism in which there are both tem-

[4]See Hocutt (1972). Hintikka (1986) also mentions that Chomsky makes the same point in Chomsky (1982).

poral and epistemic dimensions of reasoning. But the difference is that in the formalism the temporal dimension is not to reason the time of deduction taken by the agent but the external changes happening in the environment or the knower such as actions being made or new information accepted. The knowledge modality in Temporal Epistemic logic is still a purely possible world semantics defined modality and hence logically omniscient. The same thing can also be said of the epistemic modality in Dynamic Epistemic Logic.

Some unrealistic assumptions are still made in our framework. For example, we assume that agents are able to make all the deduction steps that they haven't made within a single unit of time, and suggest that an S4 or a tS4 agent is able to know that they know a formula in one step of positive introspection. Hence adjustments on the framework might be needed according to practical considerations. We also assume that agents employ axiomatic-like reasoning systems to deduce their knowledge, but with respect to the human capacity, a natural deduction type of reasoning might be preferable. However, the primary purpose of this chapter is to make a point that knowledge is dynamic in nature and without this dynamic feature revealed, it is impossible to have an epistemic logical system free from the problem of logical omniscience. Our introduction of tMEL logics just makes a case that an epistemic framework with the temporal feature of knowledge manifested can be developed. Once the temporal dimension of knowledge is made explicit, we can readily distinguish what is easily to know from what difficult, and without this distinction, the solutions proposed so far either do not solve the problem of logical omniscience, or do not model agent with full logical reasoning ability. So if the logic omniscience problem is taken seriously, we are suggesting the right direction for the future studies in epistemic logic.

Chapter 5

Non-Circular Proofs and Proof Realization in Modal Logic

5.1 Introduction

One of many applications of modal logic in computer science is the capacity to serve as logic of knowledge, helping to reason about the information transmissions in distributed systems (e.g. Parikh and Ramanujam (1985); Fischer and Immerman (1986); Halpern and Moses (1990)), and about the intentional level of multiagent systems in general (Fagin et al. (1995a); Wooldridge (2009)). Artemov's Logic of Proofs, LP (1995; 2001), later developing into Justification Logic (Fitting (2005a,b); Artemov (2006, 2008)), enhances the expressivity of modal epistemic logic by introducing justification into the language. Formulas of the like $t{:}F$ are introduced, meaning "*t is a proof of F*" or "*t is a justification of F*, " where t is a structural object, called *proof term* or *proof polynomial*, representing an explicit proof in formal arithmetic, or a justificatory entity. One of the main theorems concerning LP is in regarding its formal relation with the modal logic S4. The *realization theorem* says that S4 theorems are exactly the formulas which can be turned into LP theorems by substituting suitable proof terms for the modal occurrences. Interpreted epistemically, the theorem shows that there is indeed justification structure embedded in S4, as logic of knowledge, which can only be explicitly disclosed in the formalism of LP. The realization theorem is also the motivation for the introduction of Logic of Proofs. As a long standing question concerning the arithmetic foundation of intuitionistic logic, Gödel took the first step to embed intuitionistic logic into S4, as logic of provability (Gödel (1933)), and Artemov furnished LP with a formal arithmetic semantics and then showed the realization theorem to complete the project.

Accordingly, a constructive syntactical proof for the realization theorem is offering an algorithmic procedure to extract the reasoning processes, the justifications, from the logic of knowledge S4, and hence worth our further attention; and we find it is interesting and puzzling in the original procedure given in Artemov (1995, 2001) and later improved in Brezhnev and Kuznets (2006), that it makes a detour to analyze cut-free Gentzen style S4 proofs, even though originally LP is introduced in Hilbert style and presented in a way that it is almost a realized counterpart of the standard Hilbert style S4 system; and proof terms, which are also suggested to be regarded as combinators in some general way (Artemov (2001)), are best understood as encoding proofs in Hilbert style. So naturally, questions are raised: What happens to the Hilbert style S4 proofs? What is the formal relation between S4 proofs and LP proofs, if both in the style of Hilbert? Can we extend the result of the realization theorem to concern S4 proofs, instead just of theorems? Thus although the realization theorem is introduced with applicational importance, it seems to suggest a deeper insight of the proof structure of modal logic. One of the contributions of this chapter is to determine a complete proper subclass of Hilbert style proofs of S4, called non-circular, and show that this is exactly the class of proofs which can be realized to LP proofs.

We will first give a characterization of non-circular S4 proofs and show that the class is complete in the sense that every S4 theorem has a non-circular proof. It is our long-term goal to find an algorithm which can turn circular proofs directly into non-circular. But, partly since a proof-theoretical tool like cut elimination and normalization, which can generates normal form for Hilbert style proofs, is not available yet, right now we present something different. For, as we know, there is a natural way of translating proofs in Gentzen style to Hilbert style, we will show that, following the translation, the Hilbert style proofs obtained from cut-free proofs are non-circular. This result also gives us a hint as to why the detour takes place and why the original proof for the realization theorem works well.

In the course of the discussions, we will introduce a new logical framework $S4^{\Delta}$ to accomplish the above result. $S4^{\Delta}$ has numerical labels for each modal occurrence, which are designed to detect the non-circularity of S4 proofs. We will show that non-circular S4 proofs are precisely those that can turn into $S4^{\Delta}$ proofs by getting suitable numerical labels. Once we have $S4^{\Delta}$ proofs, we can translate them into LP proofs quite efficiently, and furthermore, we will show that every LP proof is obtained through such a translation. Putting all these ingredients together, we have the main proof realization result connecting non-circular S4 proofs and LP proofs, and the overall procedure also provides an alternative algorithm for the realization between theorems of S4 and LP.

We can view S4$^{\Delta}$ as an immediate logic between S4 and LP. It is a useful tool, as we can see later, for the study of the structures of the logics on the both sides; but it is also an interesting logic worth studying for its own sake. It could be understood as a logic reasoning about knowledge and the time that the reasoner takes to make the inferences.[1] Once we understand S4$^{\Delta}$ in this way, our work has more fruitful consequences with an epistemic interpretation.

5.2 The Systems

We begin with an introduction of the starting point of this project, that is, to establish a proof realization procedure between systems S4 and LP. S4 is a normal modal logic with language $L_{\Box}$ built up from propositional letters $\mathcal{P}$, boolean connectives $\neg, \vee, \wedge, \rightarrow$, and an unary modal operator $\Box$. The standard S4 (Hilbert style) proof system is the following:

Axiom Schemes:

A0 axiom schemes of classical propositional logic
A1 $\Box(F \rightarrow G) \rightarrow (\Box F \rightarrow \Box G)$
A2 $\Box F \rightarrow \Box(\Box F)$
A3 $\Box F \rightarrow F$

Inference rules

R1 $F, F \rightarrow G \vdash G$ "modus ponens"
R2 $\vdash \Box F$, if $\vdash F$ "necessitation"

On the other side, LP can be viewed as a multimodal logic with *proof terms Tm* as modalities. Proof terms are built up from *proof constants* $\mathcal{C}$, *proof variables* $\mathcal{X}$, and basic proof operations: *application* $\cdot$, *proof checker* ! and *indeterminate choice* $+$. The grammar for the proof terms is: $t{:}{=}c|x|t{\cdot}t|t{+}t|!t$, where $c \in \mathcal{C}$ and $x \in \mathcal{X}$, and if ϕ is a formula in the language of LP, denoted as $L_{:}$, so is $t{:}\phi$. The system of LP introduced in Artemov (2001) is the following:

Axiom Schemes:

A0 axiom schemes of classical propositional logic
A1 $s{:}(F \rightarrow G) \rightarrow (t{:}F \rightarrow (s{\cdot}t){:}G)$
A2 $s{:}F \rightarrow !s{:}(s{:}F)$
A3 $s{:}F \rightarrow F$
A4 $s{:}F \rightarrow (s{+}t){:}F$, $s{:}F \rightarrow (t{+}s){:}F$

[1]This chapter has its origin in Wang (2009b, 2011b). Following that, the logic of knowledge and time is developed in Wang (2009a, 2011a), in which S4$^{\Delta}$ is renamed as tS4 as one of the timed Modal Epistemic Logics (tMEL).

Inference rules:

R1 $F, F \to G \vdash G$ "modus ponens"

R2 $\vdash c{:}F$ for $c \in \mathcal{C}$, if $\vdash F$ and F is an axiom
"axiom necessitation"

Notice that in these systems, there is flexibility in the choice of the axiom scheme A0. Any complete classical propositional axiom schemes can be wrapped together to be A0. For the purpose of this chapter, we will assume that all the systems discussed here employ the same A0 axiom scheme, and we will call the axiom schemes other than A0 *modal axiom schemes*.

At this point we give a rough definition of *realization*, which will be formally formulated later. Of a modal formula in $L_\Box$, an LP-style formula is a realization, or sometimes called an explicit counterpart since the justifications of knowledge statements are explicitly stated, if the formula is obtained from substituting proof terms for the modal occurrences in the modal formula. So we can see that the systems S4 and LP with only few exceptions are parallel to each other: every axiom scheme and rule in S4 has an explicit counterpart in the axiom system of LP, and every axiom scheme and rule in LP is an explicit counterpart of some axiom scheme and rule in S4. In addition, we can actually remove the exceptions by introducing both variants of S4 and LP. We will denote the system with the rule of *axiom necessitation* "$\vdash\Box F$, if $\vdash F$, and F is an axiom" substituting for the R2 rule of *necessitation* in S4 as S4$'$ and the system with axiom scheme A4 "$\Box F{\to}\Box F$" adding to S4$'$ as S4$''$. For LP, we introduce the following variant ELP (Below both $o(s)$ and $\dot{o}(s)$ denote proof terms of the form of finite sum with s as its summand (e.g., $t_1{+}s{+}t_2$), and $o(s){=}s$ is possible, whereas $\dot{o}(s){=}s$ is not):

Axiom Schemes:

A0 axiom schemes of classical propositional logic

A1 $s{:}(F \to G) \to (t{:}F \to o(s{\cdot}t){:}G)$

A2 $s{:}F \to o(!s){:}(s{:}F)$

A3 $s{:}F \to F$

A4 $s{:}F \to \dot{o}(s){:}F$

Inference rules:

R1 $F, F \to G \vdash G$ "modus ponens"

R2 $\vdash o(c){:}F$ for $c \in \mathcal{C}$, if $\vdash F$ and F is an axiom
"axiom necessitation"

The system with the axiom scheme A4 "$s{:}F{\to}\dot{o}(s){:}F$" removed from ELP is called ELP$^-$, and the system with the *necessitation* rule "$\vdash o(c){:}F$ for $c{\in}\mathcal{C}$, if $\vdash F$" substituting for the R2 rule of *axiom necessitation* in ELP$^-$ is called GELP$^-$. Systems S4, S4$'$ and S4$''$ prove the same set of theorems, but systems LP, ELP, ELP$^-$ and GELP$^-$ do not. However, every S4 theorem, and hence every S4′ and S4″ theorem, can be realized to a theorem in these systems of LP variants, and there would be syntactical translations between proof terms such that theorems in one of these systems are translated into theorems in another one. The reason that we introduce the system ELP, and the notations $o(s)$ and $\dot{o}(s)$ will be clear when a proof realization procedure is formally discussed.

Now we can see there are complete parallelisms between systems S4 and GELP$^-$, between S4′ and ELP$^-$, and between S4″ and ELP, and it is natural to expect that these parallelisms can be extended to between proofs. We expect there is a line-to-line proof realization procedure, by which we mean a procedure to establish a set of proof terms such that a proof in S4, S4′ or S4″, can be turned into a proof in GELP$^-$, ELP$^-$, or ELP, respectively, and, furthermore, an axiom is turned into its corresponding explicit axiom, and a formula derived by a rule into a formula derived by its corresponding explicit rule.

However, life is not that simple. As it is shown in the following example, not all proofs in the S4-systems have this kind of straightforward line-to-line proof realization. This shows the task of establishing a procedure realizing S4 theorems to LP theorems is not as trivial as we might think at our first pass of these systems. Here's a fragment of a proof:

$\phi_1 \equiv \Box(P \to Q) \to (\Box P \to \Box Q)$
$\phi_2 \equiv \Box(Q \to P) \to (\Box Q \to \Box P)$
$\phi_3 \equiv \Box(P \to Q) \to (\Box(Q \to P) \to (\Box P \to \Box P))$
$\phi_4 \equiv \Box(P \to Q) \to (\Box(Q \to P) \to (\Box Q \to \Box Q))$
$\phi_5 \equiv \Box(P \to Q) \to (\Box(Q \to P) \to ((\Box P \to \Box P) \wedge (\Box Q \to \Box Q)))$.

To save space, the intended proof is not listed in full here. Some steps are skipped, but the proof that we have in mind is that ϕ_3 is derived from ϕ_1 and ϕ_2 through some kind of general classical syllogism such that the subformula $\Box Q$ is removed. Notice that in the realization of this part of the proof the two occurrences of $\Box Q$ must be realized to the same LP formula in order that the classical syllogism can be parallelly applied in the corresponding LP-systems. Similarly, ϕ_4 is intended to be derived from ϕ_1 and ϕ_2 by removing $\Box P$, and finally ϕ_5 is derived from ϕ_3 and ϕ_4 through classical propositional logic.

Given that ϕ_1 and ϕ_2 are instances of modal axioms, their realizations should be explicit axioms:

$\phi_1^r \equiv s{:}(P \to Q) \to t{:}P \to o(s{\cdot}t){:}Q$
$\phi_2^r \equiv u{:}(Q \to P) \to v{:}Q \to o(u{\cdot}v){:}P$

for some proof terms s, t, u, v, and since syllogisms are applied, $v{=}o(s{\cdot}t)$ and $t{=}o(u{\cdot}v)$ must be the cases. Now due to the complexity of proof terms, no solution to these equations is possible, and hence the naive proof realization procedure doesn't work for this instance. In the next section we will give the definition of non-circular proofs which is hinted at this example.

5.3 Non-Circular Proofs

Our following discussions of Hilbert style proofs will be based on the analysis of formula occurrences, and hence a formal definition of occurrences and formulas at occurrences will be helpful and make clear our arguments. We will give an abstract description of paths of the parse tree of a formula ϕ to denote the positions, *occurrences*, of its subformulas, and the function $\phi(x)$ to denote the subformula at the occurrence x. The language of occurrences $\mathcal{O}$ are sequences of letters a, b, and $\star$. The symbol . (dot) doesn't belong to the formal syntax but will sometimes be written within a sequence to increase readability. We use a, b to denote the left and right positions of a binary operator, and $\star$ to denote the position of the operand of an unary operator. $\circ$ denotes a metavariable for a connective.

Definition 5.3.1 (occurrence)**.**
Let $\phi \in L_\Box$. We define simultaneously the set of occurrences in ϕ, $\mathcal{O}(\phi)$, and the function $\phi(\cdot)$ which maps an occurrence in ϕ to the subformula of ϕ at the occurrence. Let ϵ be the empty sequence.

1. *$\epsilon \in \mathcal{O}(\phi)$ and $\phi(\epsilon) = \phi$,*
2. *If $x \in \mathcal{O}(\phi)$ and $\phi(x) = (\psi \circ \theta)$, then $x.a, x.b \in \mathcal{O}(\phi)$ and $\phi(x.a) = \psi$, and $\phi(x.b) = \theta$,*
3. *if $x \in \mathcal{O}(\phi)$ and $\phi(x) = (\circ\psi)$, then $x.\star \in \mathcal{O}(\phi)$ and $\phi(x.\star) = \psi$,*
4. *Furthermore, we extend the definition on formulas to sequences of formulas, such as proofs. Let $\mathcal{D}$ be a sequence of n formulas in $L_\Box$ and ϕ be the k-th element of $\mathcal{D}$, then $i \in \mathcal{O}(\mathcal{D})$ for any $i \leq n$ and $\mathcal{D}(k) = \phi$.*

Thus suppose $(A \to \Box B) \to A$ is the second element of a proof $\mathcal{D}$, then $2ab.\star \in \mathcal{O}(\mathcal{D})$ and $\mathcal{D}(2ab.\star) = B$.

Here are some facts about the notion of occurrence ($x, y, z \in \mathcal{O}$):

- $\epsilon.x \equiv x \equiv x.\epsilon$,
- if $x \in \mathcal{O}(\phi)$ and $y \in \mathcal{O}(\phi(x))$, then $x.y \in \mathcal{O}(\phi)$, and $\phi(x.y) = \phi(x)(y)$,
- if $\phi(x) = \phi(y)$, then $x.z \in \mathcal{O}(\phi)$ iff $y.z \in \mathcal{O}(\phi)$,
- if ρ is a propositional letter substitution, and ϕ^ρ is the result of the substitution, then $\mathcal{O}(\phi) \subseteq \mathcal{O}(\phi^\rho)$.

A Hilbert style proof is a sequence of formulas which are either axiom instances or derived by rule applications. But this also implies that subformulas at some of the occurrences are mandatory to be equal (in order for a formula in the sequence to be an axiom or derived by a rule), and some are not. In the following, the definition of *proof equivalence relation* is trying to capture this idea. We call formulas of the form $\Box\phi$ *m-formulas*, and $m(\mathcal{D})$ is used to denote the set of occurrences of m-subformulas in $\mathcal{D}$. In this chapter, we only care about the relation between m-subformula occurrences in a proof. We will use a *label function* to supply labels for m-formula occurrences in order to relate them. An m-formula $\Box P$ is atomic if $P \in \mathcal{P}$. Two atomic m-subformula occurrences are related to each other if they get the same label, and two general subformula occurrences are related to each other if all their m-subformula occurrences get the same labels. Here are the formal definitions.

Definition 5.3.2 (proof equivalence relation)**.**
Let $\mathcal{D}$ be a proof in S4, S4′ or S4″, and k a natural number. An equivalence relation $\sim$ on $\mathcal{O}(\mathcal{D})$ is called a proof equivalence relation *if it satisfies:*

1. *if $\mathcal{D}(k)=A^\rho$ with A an axiom scheme, ρ a propositional letter substitution, and $A(x)=A(y)\in\mathcal{P}$ for $x,y\in\mathcal{O}(A)$, then $k.x\sim k.y$,*[2]
2. *if $\mathcal{D}(k)$ is a substitutional instance of the axiom scheme A2, i.e. $\mathcal{D}(k)=\Box\phi\to\Box(\Box\phi)$, then furthermore $k.a\sim k.b\star$,*
3. *if $\mathcal{D}(k)=\psi$ is derived from $\mathcal{D}(i)=\phi$ and $\mathcal{D}(j)=\phi\to\psi$ by modus ponens, then $k\sim j.b$ and $i\sim j.a$,*
4. *if $\mathcal{D}(k)=\Box\phi$ is derived from $\mathcal{D}(i)=\phi$ by necessitation (in S4) or axiom necessitation (in S4′ or S4″), then $k.\star\sim i$.*

Definition 5.3.3. *(label function)*
Given $\mathcal{D}$ a formula or a sequence of formulas in $L_\Box$, and $\mathbf{I}$ a label set, we call $l\colon m(\mathcal{D})\to\mathbf{I}$ a label function *on $\mathcal{D}$. Any nonempty set can be a label set.*

Each label function l on $\mathcal{D}$ induces an equivalence relation $\overset{l}{\sim}$ on $\mathcal{O}(\mathcal{D})$ such that for any $x,y\in\mathcal{O}(\mathcal{D})$, $x\overset{l}{\sim}y$ iff $\mathcal{D}(x)=\mathcal{D}(y)$, and for any $x.z$, $y.z\in m(\mathcal{D})$, $l(x.z)=l(y.z)$.

Definition 5.3.4 (proof label function)**.**
A label function l on a proof $\mathcal{D}$ in S4, S4′ or S4″ is a proof label function *if the induced equivalence relation $\overset{l}{\sim}$ is a proof equivalence relation.*

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[2] We equate axiom schemes with their simplest propositional letter substitutional instances.

Definition 5.3.5. *For two label functions l, l' on $\mathcal{D}$, we say l' covers l if for any x, y in $\mathcal{O}(\mathcal{D})$,$x \overset{l'}{\sim} y$, whenever $x \overset{l}{\sim} y$.*

Lemma 5.3.6. *If l is a proof label function on a proof $\mathcal{D}$, and l' is a label function on $\mathcal{D}$ such that l' covers l, then l' is also a proof label function on $\mathcal{D}$.*

Lemma 5.3.7. *Given a proof $\mathcal{D}$, and label functions l, l' on $\mathcal{D}$, if for any $x, y \in m(\mathcal{D})$ with $x \overset{l}{\sim} y$, $l'(x) = l'(y)$, then l' covers l.*

According to the definition, a proof can be supplied with more than one proof label function. A proof label function can be as coarse as the label function which assigns the same label to every modal occurrence. But the finer the proof label function with respect to the relation of covering, the more essence of the structure of the proof preserved in the proof label function.

As our earlier observation of the unrealizable proof fragment shows, what matters is some specific relations among m-formula occurrences in modal axioms. We will call the collection of the relations in concern the *stamp* of the modal logical system.

Definition 5.3.8 (the standard stamp of S4)**.**

A stamp $\mathcal{A}$ of a modal logical system is a collection of binary relations $\overset{A}{\to}$ on $m(A)$ with A a modal axiom scheme.

The standard stamp of S4 and S4′ include (the scheme names stand for the schemes):

$\overset{A1}{\longrightarrow} = \{ \langle a, b.b \rangle, \langle b.a, b.b \rangle \}$,

$\overset{A2}{\longrightarrow} = \{ \langle a, b \rangle \}$,

and, one more for S4″:

$\overset{A4}{\longrightarrow} = \{ \langle a, b \rangle \}$.

That is, the standard stamp is concerned with the directed relations from $\Box(F \to G)$ to $\Box G$ and from $\Box F$ to $\Box G$ in the axiom scheme $\Box(F \to G) \to (\Box F \to \Box G)$, from $\Box F$ to $\Box(\Box F)$ in $\Box F \to \Box\Box F$, and from the first $\Box F$ to the second $\Box F$ in $\Box F \to \Box F$.

Below $[x]^l$ denotes the equivalence class induced by the label function l and containing the occurrence x.

Definition 5.3.9. *Given a proof label function l on $\mathcal{D}$ in a system with stamp $\mathcal{A}$, $\overset{\mathcal{A}}{\to}$ is a relation defined on the equivalence classes induced by l such that if $\mathcal{D}(k)$ is an axiom instance of an axiom scheme A and $x \overset{A}{\to} y$ with $\overset{A}{\to} \in \mathcal{A}$, then $[k.x]^l \overset{\mathcal{A}}{\to} [k.y]^l$.*

A chain of l-induced equivalence classes, E_1, E_2, ..., E_n, with respect to the relation $\xrightarrow{\mathcal{A}}$, namely, $E_i \xrightarrow{\mathcal{A}} E_{i+1}$ for any E_i in the chain, is circular, if there are $i \neq j$, $E_i = E_j$.

Now we give the formal definition of non-circular proof.

Definition 5.3.10 (non-circular proof)**.**

A proof label function l on a proof $\mathcal{D}$ in a system with stamp $\mathcal{A}$ is $\mathcal{A}$-circular if there is a chain of l-induced equivalence classes with respect to $\xrightarrow{\mathcal{A}}$ which is circular; otherwise l is $\mathcal{A}$-non-circular.

A proof $\mathcal{D}$ is $\mathcal{A}$-non-circular if we can define an $\mathcal{A}$-non-circular proof label function on $\mathcal{D}$.

So given a modal logical system, such as S4, there is more than one stamp that can be defined. The standard stamp that we are going to discuss is not the only one. But what is interesting about this stamp, and others for S4′ and S4″, is that we can be sure that the classes of non-circular proofs with respect to these stamps are complete. A proof of this result is the aim of the next section.

We will skip mentioning the stamps or what are the stamps $\mathcal{A}$ referring to in the ensuing discussions, which are presumed to be the standard as defined in Definition 5.3.8.

5.4 S4$^{\Delta}$ and the Completeness of Non-Circular Proofs

5.4.1 S4 and S4$^{\Delta}$

The main goal of this section is to establish the completeness of non-circular proofs, and for this purpose, we introduce logical systems S4$^{\Delta}$, and its variants. There are several reasons for this introduction. First of all, in the above section we give a general definition of non-circular proof with the label function as an auxiliary tool. The non-circularity of proofs essentially depends on the relations between m-formulas in the proof. However, it is possible to employ a structural label set to detect the non-circularity of proofs, and the set of natural number is a good candidate. Secondly, it is always easier to work on formulas with their labels built in as part of the formulas, instead of to work on formula occurrences and then consider their labels. Finally, since S4$^{\Delta}$ is introduced as a logical system, hence logical techniques can be applied to deal with the relevant problems. We need more properties of proof label function.

Definition 5.4.1 (increasing proof label function)**.**

A numerical proof label function $\Delta\colon m(\mathcal{D})\to\mathbb{N}$ *on a proof* $\mathcal{D}$ *in S4, S4′, or S4″ is increasing if* $\Delta(k.x) < \Delta(k.y)$*, for any substitutional instance* $\mathcal{D}(k)$ *of an axiom scheme* A*, and any occurrences* $x \overset{A}{\to} y$ *with* $\overset{A}{\to}\in \mathcal{A}$*.*

Lemma 5.4.2. Δ *is an increasing proof label function on a proof* $\mathcal{D}$*, if and only if for any* $x, y \in \mathcal{O}(\mathcal{D})$*, if* $[x]^{\Delta} \overset{\mathcal{A}}{\to} [y]^{\Delta}$*, then* $\Delta(x) < \Delta(y)$*.*

Proof. By the definitions of increasing proof label function and the fact that if $w \overset{\Delta}{\sim} z$, then $\Delta(w) = \Delta(z)$. ⊣

Proposition 5.4.3. *A proof* $\mathcal{D}$ *is non-circular if and only if there exists an increasing proof label function* Δ *on* $\mathcal{D}$*.*

Proof. When Δ is increasing, by the Lemma 5.4.2, it immediately follows that every chain of Δ-induced equivalence classes with respect to $\overset{\mathcal{A}}{\to}$ is non-circular. Hence every increasing proof label function is non-circular. On the other hand, when $\mathcal{D}$ is non-circular, there exists a proof label function l on $\mathcal{D}$ such that every chain of l-induced equivalence classes with respect to $\overset{\mathcal{A}}{\to}$ is non-circular. Let S be the set of these chains. S will be a finite set of finite chains. We define the function $\Delta_l\colon m(\mathcal{D})\to\mathbb{N}$ such that $\Delta_l(x) = \max\{i \mid [x]^l \text{ is the } i\text{-th element of a chain in } S\}$. Since $\Delta_l(x)=\Delta_l(y)$ for any $x\overset{l}{\sim}y$ in $m(\mathcal{D})$, Δ_l covers l and hence Δ_l is a proof label function. Also, since $\Delta_l(x)<\Delta_l(y)$ for any $[x]^l\overset{\mathcal{A}}{\to}[y]^l$, Δ_l is increasing. This completes the proof. ⊣

Given a non-circular proof label function l on a proof $\mathcal{D}$, we can actually build an increasing proof label function such that different *initial equivalence classes*, equivalence classes without predecessors, have different number labels.

Definition 5.4.4. *An equivalence class* $[x]^l$ *is* initial *if and only if there's no* $[y]^l$ *such that* $[y]^l \overset{\mathcal{A}}{\to} [x]^l$*.*

Corollary 5.4.5. *Let* S *be the set of initial equivalence classes in* $\mathcal{D}$ *and* $f\colon S\to\mathbb{N}$*. There exists an increasing proof label function* Δ *covering* l *such that for any* $[x]^l\in S$*,* $\Delta(x)=f([x]^l)$*.*

Proof. Let $d=\max\{f([x]^l) \mid [x]^l\in S\}$, and Δ_l be the increasing function built based on the procedure given in the above Proposition 5.4.3. Then the numerical function Δ such that for any $x\in m(\mathcal{D})$, $\Delta(x)=f([x]^l)$ if $[x]^l$ is initial, otherwise $\Delta(x)=\Delta_l(x)+d$ will do the job. ⊣

Now we introduce a new family of languages. Let $\mathbf{I}$ be a label set. The language $L_{\mathbf{I}}$ is an extended propositional language with the following non-propositional formula formation rule: if $\phi \in L_{\mathbf{I}}$ and $u \in \mathbf{I}$, $\Box\phi^u \in L_{\mathbf{I}}$. We will also call a formula of the form $\Box\phi^u$ m-formula, and call the label u the principal label of the formula, denoted as $\nu(\Box\phi^u)$.

The Definition 5.3.1 can be well-adapted to define formula occurrences of formulas or sequences of formulas in $L_{\mathbf{I}}$. The only needed change is to take care of the new m-formulas: if $x \in \mathcal{O}(\phi)$ and $\phi(x)$=$(\Box\psi^u)$, then $x.\star \in\mathcal{O}(\phi)$ and $\phi(x.\star)$=ψ.

Later in this chapter we will see several translations between formulas and proofs in $L_\Box$ and $L_{\mathbf{I}}$. If not stated otherwise, they are all presumed to *fix propositional letters* and *commute with boolean connectives*. In other words, the purpose of these translations is to add or remove labels, or to switch the labels from one to another. In fact, we can view a realization as a translation of this kind with proof terms as labels.

Let $\mathcal{D}$ be a sequence of formulas in $L_\Box$ and $\mathcal{F}$ be a sequence of formulas in $L_{\mathbf{I}}$ with $\mathbf{I}$ a label set.

Definition 5.4.6.

For any label function $l\colon m(\mathcal{D}) \to \mathbf{I}$, an induced translation, also denoted as l, from $\mathcal{D}$ to a sequence of formulas $\mathcal{D}^l$ in $L_{\mathbf{I}}$ is such that for any $x \in m(\mathcal{D})$, $\mathcal{D}^l(x) = \Box\mathcal{D}^l(x.\star)^u$ with $u = l(x)$.

A $\Box$-translation on $\mathcal{F}$ is a translation such that for every $x \in m(\mathcal{F})$, $\mathcal{F}^\Box(x) = \Box\mathcal{F}^\Box(x.\star)$,

$l_\mathcal{F}$ is a label function on $\mathcal{F}^\Box$ induced by $\mathcal{F}$ such that for every $x \in m(\mathcal{F}^\Box)$, $l_\mathcal{F}(x) = \nu(\mathcal{F}(x))$.

$l_\mathcal{F}$ is well-defined since $x{\in}m(\mathcal{F}^\Box)$ if and only if $x{\in}m(\mathcal{F})$, and we have the following equivalence results:

Lemma 5.4.7. *For x, $y \in \mathcal{O}(\mathcal{D})$, $x \overset{l}{\sim} y$ iff $\mathcal{D}^l(x) = \mathcal{D}^l(y)$.*

Proposition 5.4.8. *$\mathcal{F} = \mathcal{D}^l$ iff $\mathcal{F}^\Box = \mathcal{D}$ and $l_\mathcal{F} = l$.*

S4$^\Delta$ is a logical system defined on the set of formulas L_Δ, a case of $L_{\mathbf{I}}$ languages with natural numbers as labels. The system is the following:

Axiom Schemes:

A0 axiom schemes of classical propositional logic

A1 $\Box(F \to G)^i \to (\Box F^j \to \Box G^k)$, $i, j < k$

A2 $\Box F^i \to \Box(\Box F^i)^j$, $i < j$

A3 $\Box F^i \to F$

Inference rules

R1 $F, F \to G \vdash G$ "modus ponens"

R2 $\vdash \Box F^i$ for any i, if $\vdash F$ "necessitation"

System $\mathrm{S4}'^{\Delta}$ is $\mathrm{S4}^{\Delta}$ with *necessitation* replaced by *axiom necessitation* "$\vdash\Box F^i$ for any i, if $\vdash F$ and F is an axiom," and $\mathrm{S4}''^{\Delta}$ is $\mathrm{S4}'^{\Delta}$ with the addition of the axiom scheme A4, "$\Box F^i \to \Box F^j, i<j$."

An interesting and apparent feature of the system is that for a formula being an axiom instance, the number labels in the formula have to satisfy some condition, and this feature is just the key to the the success of the following theorem concerning the formal relations between proofs in variant S4-systems and $\mathrm{S4}^{\Delta}$-systems.

Theorem 5.4.9. *A proof $\mathcal{D}$ in S4, S4′, or S4″ is non-circular if and only if there is a proof label function $\Delta\colon m(\mathcal{D}) \to \mathbb{N}$ such that $\mathcal{D}^{\Delta}$ is a proof in $S4^{\Delta}$, $S4'^{\Delta}$, or $S4''^{\Delta}$, respectively.*

Proof. Given $\mathcal{D}^{\Delta}=\mathcal{F}$ a proof in $\mathrm{S4}^{\Delta}$ [$\mathrm{S4}'^{\Delta}$, $\mathrm{S4}''^{\Delta}$], it is not difficult to check that $\mathcal{D}$ ($=\mathcal{F}^{\Box}$) is a proof in S4 [S4′, S4″] and that Δ ($=l_{\mathcal{F}}$) is an increasing proof label function on $\mathcal{D}$ since it needs to satisfy the numerical conditions set on the modal axiom schemes of the system. Hence $\mathcal{D}$ is non-circular. For the other direction, suppose that $\mathcal{D}$ is a non-circular proof in $\mathrm{S4}^{\Delta}$ [$\mathrm{S4}'^{\Delta}$, $\mathrm{S4}''^{\Delta}$]. By Proposition 5.4.3, an increasing proof label function Δ defined on $\mathcal{D}$ exists, and hence all we need to do is to check if $\mathcal{D}^{\Delta}$ is a proof in $\mathrm{S4}^{\Delta}$ [$\mathrm{S4}'^{\Delta}$, $\mathrm{S4}''^{\Delta}$]. Since Δ is a proof label function, then it can be sure by Lemma 5.4.7 that whenever ϕ is an axiom or derived by a rule application in $\mathcal{D}$, so is ϕ^{Δ} in $\mathcal{D}^{\Delta}$ except that ϕ is a modal axiom. But since Δ is also increasing, conditions on modal axioms will be fulfilled, and hence $\mathcal{D}^{\Delta}$ is a proof in $\mathrm{S4}^{\Delta}$ [$\mathrm{S4}'^{\Delta}$, $\mathrm{S4}''^{\Delta}$]. ⊣

Corollary 5.4.10. *$\mathcal{F}=\mathcal{D}^{\Delta}$ is a proof in $S4^{\Delta}$, $S4'^{\Delta}$, or $S4''^{\Delta}$, if and only if Δ ($=l_{\mathcal{F}}$) is an increasing proof label function on $\mathcal{D}$ ($=\mathcal{F}^{\Box}$).*

5.4.2 Completeness of Non-Circular Proofs

As mentioned in the introduction, the long-term goal of this project is to establish a direct procedure turning Hilbert style circular proofs into non-circular, which the completeness of non-circular proofs immediately follows. Right now we will deal with the problem by analyzing Gentzen style proofs. We first supply the Gentzen systems that corresponds to S4 and $\mathrm{S4}^{\Delta}$, respectively. Here are some notations. A sequent $\Gamma \Rightarrow \Gamma'$ is a pair of finite multisets Γ, Γ' of formulas. It is convenient for us to view a sequent as a formula $C_1 \to (\ldots \to (C_n \to \bigvee \Gamma') \ldots)$. Given a multiset $\Gamma = \{C_i\}$ of formulas in

$L_\Box$, $\Box\Gamma=\{\Box C_i\}$. Given a multiset $\Gamma=\{C_i\}$ of formulas in L_Δ, $\Box\Gamma^\iota=\{\Box C_i^{j_i}\}$, for j_i a number in the multiset ι. $|\Gamma|$ is the number of formulas in Γ. The Gentzen system S4G is:

The only axiom is that $P \Rightarrow P$, for a propositional letter P.
The rules for weakening (W) and contraction (C)

$$\text{LW}\ \frac{\Gamma \Rightarrow \Gamma'}{A, \Gamma \Rightarrow \Gamma'}\ ,\quad \text{RW}\ \frac{\Gamma \Rightarrow \Gamma'}{\Gamma \Rightarrow \Gamma', A}$$

$$\text{LC}\ \frac{A, A, \Gamma \Rightarrow \Gamma'}{A, \Gamma \Rightarrow \Gamma'}\ ,\quad \text{RC}\ \frac{\Gamma \Rightarrow \Gamma', A, A}{\Gamma \Rightarrow \Gamma', A}$$

The classical logical rules (i=0,1):

$$\text{L}\neg\ \frac{\Gamma \Rightarrow \Gamma', A}{\neg A, \Gamma \Rightarrow \Gamma'}\ ,\quad \text{R}\neg\ \frac{\Gamma, A \Rightarrow \Gamma'}{\Gamma \Rightarrow \Gamma', \neg A}$$

$$\text{L}\wedge\ \frac{A_i, \Gamma \Rightarrow \Gamma'}{A_0 \wedge A_1, \Gamma \Rightarrow \Gamma'}\ ,\quad \text{R}\wedge\ \frac{\Gamma \Rightarrow \Gamma', A \quad \Gamma \Rightarrow \Gamma', B}{\Gamma \Rightarrow \Gamma', A \wedge B}$$

$$\text{L}\vee\ \frac{A, \Gamma \Rightarrow \Gamma' \quad B, \Gamma \Rightarrow \Gamma'}{A \vee B, \Gamma \Rightarrow \Gamma'}\ ,\quad \text{R}\vee\ \frac{\Gamma \Rightarrow \Gamma', A_i}{\Gamma \Rightarrow \Gamma', A_0 \vee A_1}$$

$$\text{L}\rightarrow\ \frac{\Gamma \Rightarrow \Gamma', A \quad B, \Gamma \Rightarrow \Gamma'}{A \rightarrow B, \Gamma \Rightarrow \Gamma'}\ ,\quad \text{R}\rightarrow\ \frac{A, \Gamma \Rightarrow \Gamma', B}{\Gamma \Rightarrow \Gamma', A \rightarrow B}$$

The modal rules:

$$\text{L}\Box\ \frac{A, \Gamma \Rightarrow \Gamma'}{\Box A, \Gamma \Rightarrow \Gamma'}\ ,\quad \text{R}\Box\ \frac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ .$$

S4G as listed above is similar to the propositional fragment of **G1s** in Troelstra and Schwichtenberg (1996), except that it is a system for the language with single modality $\Box$, and with the negation $\neg$ instead of the falsehood $\bot$. It is therefore complete with respect to the standard Hilbert style system of S4.

S4$^\Delta$G is a Gentzen style proof system defined on formulas in L_Δ. Its axiom and rules are the same as the ones in S4G except that its modal rules are the following:

$$\text{L}\Box\ \frac{A, \Gamma \Rightarrow \Gamma'}{\Box A^i, \Gamma \Rightarrow \Gamma'}\ ,\ \text{for any } i$$

$$\text{R}\Box\ \frac{\Box\Gamma^\iota \Rightarrow A}{\Box\Gamma^\iota \Rightarrow \Box A^i}\ ,\ \text{for any } i > \max(\iota) + |\Gamma|, \text{ when } |\Gamma| \neq 0, \text{ and for any } i \text{ when } |\Gamma| = 0.$$

There are two main steps in our procedure of proving the completeness of non-circular proofs. The first step is to show that every S4G proof can be turned into an S4$^\Delta$G proof, and the second is that S4$^\Delta$G is sound with respect to S4$^\Delta$. In the following, when we adjust m-formula occurrences' number labels in an S4$^\Delta$G proof, we adjust all the related formulas of premises and conclusions of rules to the same number. Obviously S4G is a cut-free system, and recall that cut-free proofs respect the polarity of formulas.

Lemma 5.4.11. *If in an $S4^{\Delta}G$ proof we adjust the number labels such that the principal labels of negative m-formula occurrences become smaller, and those of positive m-formula occurrences become larger, the result will still be an $S4^{\Delta}G$ proof.*

Proof. The only applications of inference rules will be affected are the applications of the right modal rule. However, the numerical condition on the rule is still fulfilled after the adjustment. ⊣

Proposition 5.4.12. *Every S4G proof $\mathcal{G}$ can be translated to a proof $\mathcal{G}^{\Delta}$ in $S4^{\Delta}G$ by providing suitable numerical labels for m-formula occurrences in $\mathcal{G}$.*

Proof. The proof is quite straightforward. We can give suitable labels to an S4G proof by induction on the depth of the prooftree. There are some cases, like applications of two-premise inference rules, in which the labels need adjustments. In these cases, we can apply the previous lemma to adjust the number labels in the premises of an application and the prooftrees above the premises such that the labels of m-formulas in the premises which relate to the same formula in the conclusion match to each other. Since S4G is cut-free, so this always can be done, and then the two-premise inference rules of $S4^{\Delta}G$ can be well applied.

Nevertheless, there exists a very efficient method. We can just let all negative formula occurrences have the label 0, and all positive formula occurrences have the label equal to the number of m-formula occurrences in the S4G proof. Then the numerical conditions on all the applications of the right modal rules will be satisfied. ⊣

Notice that the efficient method mentioned above won't work when the modal rule have conditions such that the principal label of a positive m-formula relies on the principal labels of other positive m-formulas, like the Gentzen system for S5, which we will discuss later.

Proposition 5.4.13. *Every $S4^{\Delta}G$ proof can be converted to an $S4^{\Delta}$ proof with the same conclusion.*

Proof. The procedure is to convert each application of an inference rule (including axioms) to a sequence of formulas. For the propositional part, we can pick up the procedure listed in Cook and Reckhow (1974) and for the applications of the left modal rule L□, the translation is not difficult to figure out. Here we only check that there is such a conversion for applications of the right modal rules, R□. We need the following lemma:

Lemma 5.4.14. *For* $|\Gamma| > 0$ *and* $i > \max(\max(\iota)+1, e) + |\Gamma| - 1$, $\Box(\Box\Gamma^{\iota} \Rightarrow A)^{e} \to (\Box\Gamma^{\iota} \Rightarrow \Box A^{i})$ *is provable in* $S4^{\Delta}$.

Proof. It's equivalent to prove that for any $|\Theta| \geq 0$ and $i > \max(\max(\iota, j) + 1, e) + |\Theta|$,
$(*)$ $\Box(\Box C^j \to (\Box\Theta^\iota \Rightarrow\ A))^e \to (\Box C^j \to (\Box\Theta^\iota \Rightarrow \Box A^i))$
is provable in S4$^\varDelta$. We will prove this by induction on $|\Theta|$. Noticed that for any multiset Θ, if number $e' > max(e, j+1)$, then
$\Box(\Box C^j \to (\Box\Theta^\iota \Rightarrow A))^e \to (\Box(\Box C^j)^{j+1} \to \Box(\Box\Theta^\iota \Rightarrow A)^{e'})$
is an A1 axiom, and
$\Box C^j \to \Box(\Box C^j)^{j+1}$ is an A2 axiom, and therefore
$(**)$ $\Box(\Box C^j \to (\Box\Theta^\iota \Rightarrow\ A))^e \to (\Box C^j \to \Box(\Box\Theta^\iota \Rightarrow A)^{e'})$
is provable in S4$^\varDelta$.

When Θ is empty, let $e' = i > \max(e, j+1)$. Then $(**)$ holds, and hence the base case of $(*)$ is proved. For the induction step, suppose $|\Theta| = k+1$ and $\Box\Theta^\iota = \Box\Theta'^{\iota'} \cup \Box C'^{j'}$. Let $e' = \max(j+1, e) + 1$, and hence $(**)$ holds. Now since $i > \max(\max(\iota, j) + 1, e) + |\Theta| \geq \max(\max(\iota', j') + 1, e') + |\Theta'|$, by Induction Hypothesis, $\Box(\Box\Theta^\iota \Rightarrow\ A)^{e'} \to (\Box\Theta^\iota \Rightarrow \Box A^i)$, which is equivalent to $\Box(\Box C'^{j'} \to (\Box\Theta'^{\iota'} \Rightarrow\ A))^{e'} \to (\Box C'^{j'} \to (\Box\Theta'^{\iota'} \Rightarrow \Box A^i))$, holds. Then by classical propositional logic, $(*)$ is provable in S4$^\varDelta$. This finishes the induction step and the proof. ⊣

Since if $\Box\Gamma^\iota \Rightarrow A$ is provable in S4$^\varDelta$, when Γ is empty, by *necessitation*, $\Rightarrow\Box A^i$ is provable for any i, and when Γ is not empty, $\Box(\Box\Gamma^\iota \Rightarrow A)^0$ is provable, and then, following the procedure in the previous lemma, we can produce an S4$^\varDelta$ proof for $\Box\Gamma^\iota \Rightarrow \Box A^i$, whenever $i > \max(\iota) + |\Gamma|$. This completes the proof of Proposition 5.4.13. ⊣

Now we can prove one of few structural properties concerning Hilbert style proofs.

Theorem 5.4.15. *Every S4 theorem has a non-circular proof.*

Proof. Let ϕ be an S4 theorem. Since S4G is complete, an S4G proof $\mathcal{G}$ of ϕ exists. Then following Proposition 5.4.12, we can turn the S4G proof into an S4$^\varDelta$G proof $\mathcal{G}^\varDelta$ by assigning suitable numerical labels to m-subformulas. Now following the procedure in Proposition 5.4.13, we can translate the S4$^\varDelta$G proof into an S4$^\varDelta$ proof $\mathcal{F}$. Then $\mathcal{F}^\Box$ is a non-circular proof of ϕ. Moreover, $\mathcal{F}$ is the Hilbert style proof translated from $\mathcal{G}$ by the procedure similar to the one in Proposition 5.4.13 but without the need of concerning numerical labels. ⊣

We also have the realization theorem for S4$^\varDelta$:

Corollary 5.4.16. *For a formula $\phi \in L_\Box$, ϕ is an S4 theorem if and only if there is a numerical label function $\varDelta$ on ϕ such that $\phi^\varDelta$ is an S4$^\varDelta$ theorem.*

5.4.3 From $S4^{\Delta}$ to $S4'^{\Delta}$

The aim of this subsection is to provide an algorithm that translates $S4^{\Delta}$ proofs into $S4'^{\Delta}$ proofs. This algorithm is needed as an intermediate step for the realization theorem for LP, and also helps to establish the Δ-version of realization theorem for $S4'^{\Delta}$ and $S4''^{\Delta}$. We will provide two methods: one is called inductive and the other structural. The structural is efficient but limited to generalize.

We need to do some preliminary work. First, we will presuppose that in the $S4^{\Delta}$ proof in discussion every *R2-formula*, the formula derived by necessitation, is initial. This is the case when the proof is translated from an $S4^{\Delta}$G proof by the procedure given above. However, in general if an R2-formula $\Box\phi^i$ has predecessors, we can extend the proof by adding formulas including

$\Box\phi^0$, $\phi\to\phi$, $\Box(\phi\to\phi)^0$, $\Box(\phi\to\phi)^0\to(\Box\phi^0\to\Box\phi^i)$, $\Box\phi^0\to\Box\phi^i$,

and a proof of the tautology $\phi\to\phi$ if it is not an axiom.

Second, since now in our proof every R2-formula is initial, we can adjust the labels in the proof such that the number labels of these initial formulas have the numbers we would like them to have, as suggested by Corollary 5.4.5. In the following, given an $S4^{\Delta}$ proof, before we extend the proof to an $S4'^{\Delta}$ proof, we will firstly modify the number labels such that if $\Box\phi^i$ is an R2-formulas derived from the k-th element of the proof, then $i{=}k$ for the structural method, and $i{=}4^jk$ for the inductive method, where $\Box\phi^i$ is the conclusion of the j-th application of the necessitation rule.

We first see the inductive method. A lemma is in order. Here are some notations. Let $\mathcal{F}$ be an $S4^{\Delta}$ or $S4'^{\Delta}$ proof, and $l\colon \mathcal{F}\to\mathbb{N}$ be the length function of $\mathcal{F}$ ($l(\phi)=k$ provided ϕ is the k-th element of $\mathcal{F}$). We call $g\colon\mathcal{F}\to\mathbb{N}$ a *super-length function* on $\mathcal{F}$ if for every $\phi,\psi\in\mathcal{F}$, $0\leq g(\phi){-}l(\phi)$ and $g(\phi){-}l(\phi)\leq g(\psi){-}l(\psi)$ provided $l(\phi)\leq l(\psi)$. We call $\mathcal{F}$ *regular with respect to* g if $j\leq 4g(A)$ for any formula $\Box A^j$ dervied from A by axiom necessitation.

Lemma 5.4.17. *If $\mathcal{F}$ is an $S4'^{\Delta}$ proof regular with respect to a super-length function g, then there exists an $S4'^{\Delta}$ proof $\mathcal{F}'$ such that for every formula ϕ in $\mathcal{F}$, $\Box\phi^{4g(\phi)}$ is in $\mathcal{F}'$. Furthermore, $\mathcal{F}'$ is regular with respect a super-length function g', where for any formula $\phi\in\mathcal{F}$, $g'(\phi)=4g(\phi)$, and if ψ is the conclusion of $\mathcal{F}$, $g'(\Box\psi^{4g(\psi)})=4(g(\psi)+1)$.*

Proof. We will construct an $S4'^{\Delta}$ proof $\mathcal{F}'$ by inductively adding up to three formulas after each formula ϕ of $\mathcal{F}$ such that $\Box\phi^{4g(\phi)}$ is added. If ϕ is an axiom, then add $\Box\phi^{4g(\phi)}$. If ϕ is derived from $\psi\to\phi$ and ψ, then add formulas $\Box(\psi\to\phi)^j\to(\Box\psi^k\to\Box\phi^i)$, $\Box\psi^k\to\Box\phi^i$, $\Box\phi^i$ after ϕ with $j{=}4g(\psi\to\phi)$,

k=$4g(\psi)$, and i=$4g(\phi)$. Finally, if $\phi \equiv \Box A^j$ is derived from A by axiom necessitation, add formulas $\Box A^j \to \Box(\Box A^j)^i$, $\Box(\Box A^j)^i$ with i=$4g(\phi)$, which is larger than j since $\mathcal{F}$ is regular. Then it can be checked that $\mathcal{F}'$ is an S4$'^{\Delta}$ proof. Also, let $g' \colon \mathcal{F}' \to \mathbb{N}$ be the function such that $g'(\phi)$=$4g(\phi)$ if $\phi \in \mathcal{F}$, $g'(\psi)$=$4g(\phi) + i$ if ψ is not the conclusion of $\mathcal{F}$ and is the i-th formula to be added right after ϕ in the procedure ($i \leq 3$), and $g'(\Box\psi^{4g(\psi)}) = 4(g(\psi) + 1)$ if ψ is the conclusion of $\mathcal{F}$. Then $\mathcal{F}'$ is is regular with respect to g'. ⊣

Proposition 5.4.18. *(inductive method) Every S4$^{\Delta}$ proof $\mathcal{F}$ can be extended to be a proof in S4$'^{\Delta}$.*

Proof. Let l be the length function of $\mathcal{F}$, and $\mathcal{F}_\phi$ denote the initial segment of $\mathcal{F}$ up to ϕ. It is sufficient to prove that for every formula $\Box\phi^{4^j k}$ derived from the k-th element ϕ by the j-th application of necessitation, $\mathcal{F}_{\Box\phi^{4^j k}}$ can be extended to an S4$'^{\Delta}$ proof $\mathcal{F}_j$ of $\Box\phi^{4^j k}$ with $\mathcal{F}_j$ regular with respect to a super-length function g_j such that for any formula $\psi \in \mathcal{F}_{\Box\phi^{4^j k}}$, $g_j(\psi)$=$4^j l(\psi)$. The proof is by induction on j, and to simplify the discussion, $l(\Box\phi^{4^j k})$ is assumed to be $l(\phi)$+1. Suppose $\Box\phi^{4k}$ is derived from ϕ by the first application of necessitation, then $\mathcal{F}_\phi$ is also an S4$'^{\Delta}$ proof of ϕ and regular with respect to its length function, and hence, by Lemma 5.4.17, $\mathcal{F}_\phi$ can be extended to an S4$'^{\Delta}$ proof $\mathcal{F}_1$ of $\Box\phi^{4k}$ regular with respect to a function g_1 such that $g_1(\psi) = 4l(\psi)$ for any $\psi \in \mathcal{F}_{\Box\phi^{4k}}$. The base case is proved. Now suppose j>1 and $\Box\psi^{4^{j-1}s}$ is derived from ψ by the (j−1)-th application of necessitation. By Induction Hypothesis, there is an S4$'^{\Delta}$ proof $\mathcal{F}_{j-1}$ of $\Box\psi^{4^{j-1}s}$ regular with respect to a super-length function g_{j-1} such that $g_{j-1}(\theta)$=$4^{j-1}l(\theta)$ for any $\theta \in \mathcal{F}_{\Box\psi^{4^{j-1}k}}$. Now we can append in order formulas which are not in $\mathcal{F}_{\Box\psi^{4^{j-1}k}}$ but in $\mathcal{F}_\phi$ to $\mathcal{F}_{j-1}$ to form an S4$'^{\Delta}$ proof of ϕ regular with respect to a function g' such that $g'(\theta)$=$g_{j-1}(\theta)$ if $\theta \in \mathcal{F}_{j-1}$ and $g'(\theta)$=$4^{j-1}l(\theta)$ otherwise. Then applying Lemma 5.4.17 again, the induction step and the proof is complete. ⊣

Proposition 5.4.19. *(structural method) Every S4$^{\Delta}$ proof $\mathcal{F}$ can be extended to a proof in S4$'^{\Delta}$.*

Proof. We will lengthen the proof $\mathcal{F}$ inductively such that either the formula $\Box\phi^i$ will be added after the i-th non-conclusion formula ϕ of the proof, or if $\Box\phi^i$ is an R2-formula and ϕ is not an axiom, we can redirect it such that it is still derivable in the lengthened proof but not from ϕ by necessitation rule anymore. Then the resulting sequence is an S4$'^{\Delta}$ proof. If ϕ is an axiom, we add $\Box\phi^i$ right after ϕ. If ϕ is derived from the j-th element $\psi \to \phi$ and k-th element ψ, we add formulas $\Box(\psi \to \phi)^j \to (\Box\psi^k \to \Box\phi^i)$ and $\Box\psi^k \to \Box\phi^i$ after ϕ, and then either add $\Box\phi^i$, or not if $\Box\phi^i$ is an R2-formula in $\mathcal{F}$, such that it is derived from $\Box\psi^k \to \Box\phi^i$ and $\Box\psi^k$, which has been in the proof by induction. Finally, if $\phi \equiv \Box\psi^j$ is derived from the j-th formula ψ in $\mathcal{F}$, and now is derived from some other formulas if ψ is not an axiom, we add formulas $\Box\psi^j \to \Box(\Box\psi^j)^i$ after ϕ, and then add $\Box(\Box\psi^j)^i$, or not if it has been in $\mathcal{F}$ derived from $\Box\psi^j$ by necessitation, such that it is derived from $\Box\psi^j$ and $\Box\psi^j \to \Box(\Box\psi^j)^i$. This completes the proof. $\dashv$

The efficiency of the structural method over the inductive is suggested by the numerical labels employed in the proofs, but the former method only works for systems with the A2 transitivity axiom. One of the reasons for the introduction of the two methods is to show that our overall procedure for the realization theorem for LP, which will be completed in the next section, can be generalized to concern the LP-counterparts of other modal logics as well. In the inductive method, only the A2 axiom instances of the form that $\Box A^i \to \Box(\Box A^i)^j$ with A an axiom are used, and these instances are still realizable in the LP-counterparts of modal logical systems without the transitivity axiom.[3]

That we carefully elaborate the numerical labels in the proofs is to show that it is possible to use Δ-like logics to study the lengths of proofs. The original proof of the realization theorem for LP provided in Artemov (2001) is based on a direct translation from cut-free Gentzen style S4 proofs to LP proofs. The difficulty of such a procedure rests on the construction of suitable proof terms for modal occurrences through the analysis of the Gentzen style proof tree, especially dealing with the applications of the right modal rule. It is proved in Brezhnev and Kuznets (2006) that the original realization procedure will produce an LP proof with length exponential to the size of the initial cut-free Gentzen style proof, and can be improved such that only proofs with polynomial length are generated. Among other technical details, the improvement, where the instances of the A2 axiom of LP, $s{:}\phi \to !s{:}(s{:}\phi)$, play an important role, however, can be analyzed as with the same idea as the one in the improvement made by adopting the structure method instead of the inductive method in a procedure of generating $S4'^{\Delta}$ proofs from $S4^{\Delta}$ proofs, or in a similar but simplified procedure of generating S4′ proofs from S4 proofs.

The discussions in this subsection also imply that every non-circular proof of S4 can be extended to a non-circular proof of S4′, or S4″, a supersystem of S4′, and hence imply that the realization theorem also holds for $S4'^{\Delta}$ and $S4''^{\Delta}$.

[3]These LP-systems can found in Kuznets (2000); Fitting (2005a), in which the rule of *strong axiom necessitation*: "R2*: $\vdash c{:}F$ for $c \in \mathcal{C}$, if $\vdash F$ and F is an axiom or is inferable using R2*", instead of the rule of necessitation, is employed.

Corollary 5.4.20. *Every S4′ (S4″) theorem has a non-circular S4′ (S4″) proof, and for a formula $\phi \in L_{\Box}$, ϕ is an S4′ (S4″) theorem if and only if there is a numerical label function Δ such that ϕ^{Δ} is an $S4'^{\Delta}$ ($S4''^{\Delta}$) theorem.*

5.5 Proof Realization and the Realization Theorem for LP

5.5.1 $S4^{\Delta}$ and LP

Now we discuss the *proof realization* between proofs in the variant S4-systems and their explicit LP-counterparts, and what is left for us to do is to design a procedure that can translate between proofs in $S4^{\Delta}$-systems and proofs in their corresponding LP-systems. Besides, as we can see, with the guide of the number labels, a quite efficient procedure can be established.

Definition 5.5.1 (proof assignment)**.**

Given a sequence of formulas $\mathcal{F}$ in L_{Δ}, a proof assignment p on $\mathcal{F}$ assigns each pair (ϕ, i) for a subformula $\Box\phi^i$ in $\mathcal{F}$ a proof term.

Each proof assignment p induces a translation from $\mathcal{F}$ to a sequence of formulas in $L_{:}$ such that $(\Box\phi^i)^p{=}p(\phi,i){:}(\phi^p)$.

Here's a classification of m-formulas in a proof.

Definition 5.5.2. *Given a proof $\mathcal{F}$ in one of the $S4^{\Delta}$ systems,*

1. *if axiom $\Box(\phi{\to}\psi)^i{\to}(\Box\phi^j{\to}\Box\psi^k)$ is in $\mathcal{F}$, we call $\Box\psi^k$ A1-formula and the leading formula of the axiom, with $\Box(\phi{\to}\psi)^i$, $\Box\phi^j$, and ordered pair of formulas $\langle\Box(\phi{\to}\psi)^i, \Box\phi^j\rangle$ as its α-predecessor, β-predecessor, and A1-predecessor pair, respectively,*
2. *if axiom $\Box\phi^i{\to}\Box(\Box\phi^i)^j$ is in $\mathcal{F}$, we call $\Box(\Box\phi^i)^j$ A2-formula and the leading formula of the axiom, with $\Box\phi^j$ as its γ-predecessor,*
3. *if axiom $\Box\phi^i{\to}\Box\phi^j$ is in $\mathcal{F}$, we call $\Box\phi^j$ A4-formula and the leading formula of the axiom, with $\Box\phi^i$ as its δ-predecessor,*
4. *if $\Box\phi^i$ in $\mathcal{F}$ is derived by* necessitation *(in the case of $S4^{\Delta}$), or by* axiom necessitation *(in the case of $S4'^{\Delta}$ or $S4''^{\Delta}$), we call $\Box\phi^i$ R2-formula.*

Definition 5.5.3. *An m-formula in a proof can fall into more than one of the categories given in the previous definition. If a formula is in at most one of the above categories we say the formula is* stable*. Especially, we call an m-formula, say,* A1-stable*, if it is an A1-formula only, or does not belongs to any of the above categories.*

In the following definition, the notations of $o(s)$ and $\dot{o}(s)$ for some proof term s are what we used in the introduction of the systems ELP.

Definition 5.5.4 (characteristic proof assignment).
Each proof $\mathcal{F}$ in S4$^{\Delta}$, S4$'^{\Delta}$ or S4$''^{\Delta}$ will be associated with a system of equations $\mathbf{E}_{\mathcal{F}}$ for an unknown proof assignment p. The system consists of:

1. $p(\phi, i){=}o(p(\psi, j)\cdot p(\theta, k))$ *when* $\langle \Box\psi^j, \Box\theta^k\rangle$ *is a predecessor pair of* $\Box\phi^i$,
2. $p(\phi, i){=}o(!p(\psi, j))$ *when* $\Box\psi^j$ *is a* γ*-predecessor of* $\Box\phi^i$,
3. $p(\phi, i){=}\dot{o}(p(\psi, j))$ *when* $\Box\psi^j$ *is a* δ*-predecessor of* $\Box\phi^i$.
4. $p(\phi, i){=}o(c)$, *for some* $c \in \mathcal{C}$, *when* $\Box\phi^i$ *is an R2-formula. We will call these the R2-equations of* $\mathbf{E}_{\mathcal{F}}$.

A characteristic proof assignment *of $\mathcal{F}$ is a proof assignment which satisfies all the equations in $\mathbf{E}_{\mathcal{F}}$. And we call a characteristtic proof assignment* simple*, if it satisfies the minimal requirement of each condition, that is, for example, if $\langle \Box\psi^j, \Box\theta^k\rangle$ is a predecessor pair of $\Box\phi^i$, $p(\phi, i){=}p(\psi, j)\cdot p(\theta, k)$, and if $\Box\psi^j$ is a δ-predecessor of $\Box\phi^i$, $p(\phi, i){=}p(\psi, j){+}t$ or $p(\phi, i){=}t{+}p(\psi, j)$ for some proof term t.*

The definitions of stable formula and simple characteristic proof assignment are needed for the next subsection.

Lemma 5.5.5. *Each proof $\mathcal{F}$ in S4$^{\Delta}$, S4$'^{\Delta}$ or S4$''^{\Delta}$ has a characteristic proof assignment.*

Proof. We will construct a characteristic proof assignment by induction on the principal labels. Let i be the smallest principal label of m-formulas in $\mathcal{F}$. If $\Box\phi^i$ is an R2-formula, the pair (ϕ, i) is assigned with a proof constant c and otherwise, the pair is assigned with an arbitrary proof term.

Now suppose that for any $k{<}i$, $p(\psi, k)$ has been determined. If $\Box\phi^i$ is initial, then following the previous step, assign a proof constant or an arbitrary proof term to (ϕ, i). If $\Box\phi^i$ has only one predecessor $\Box\psi^j$ and which is a δ-predecessor with $p(\psi, j){=}s$, let $p(\phi, i){=}s{+}t$ for a term t. For the other cases, let S be the set of proof terms consisting of $s\cdot t$ if $\langle \Box\psi^j, \Box\theta^k\rangle$ is a predecessor pair of $\Box\phi^i$ with $p(\psi, j){=}s$ and $p(\theta, k){=}t$, $!s$ if $\Box\psi^j$ is a γ-predecessor with $p(\psi, j){=}s$, s if $\Box\psi^j$ is an δ-predecessor with $p(\psi, j){=}s$, and a constant c if $\Box\phi^i$ is also an R2-formula. And then let $p(\phi, i) = \sum_{s\in S} s$. By this construction, $p(\phi, i)$ certainly satisfies all the equations in $\mathbf{E}_{\mathcal{F}}$ with $p(\phi, i)$ at the left-hand side. Continue this process until every pair is assigned with some term, then p is a characteristic proof assignment. $\dashv$

Theorem 5.5.6. *$\mathcal{H}$ is a proof in GELP$^-$, ELP$^-$, or ELP if and only if there is a proof $\mathcal{F}$ in S4$^\Delta$, S4$'^\Delta$, or S4$''^\Delta$ and a characteristic proof assignment p on $\mathcal{F}$ such that $\mathcal{H} = \mathcal{F}^p$.*

Proof. Given a proof $\mathcal{F}$ in S4$^\Delta$ [S4$'^\Delta$, S4$''^\Delta$] and a characteristic proof assignment p on $\mathcal{F}$, it's straightforward to check that $\mathcal{H} = \mathcal{F}^p$ is a proof in GELP$^-$ [ELP$^-$, ELP]. For the other direction, let $Tm(\mathcal{H})$ be the set of proof terms in $\mathcal{H}$, and $\eta\colon Tm(\mathcal{H}) \to \mathbb{N}$ be an injective function linearizing the subterm relation on $Tm(\mathcal{H})$. Then for the translation induced by η such that $(t{:}\phi)^\eta = \Box(\phi^\eta)^{\eta(t)}$, $\mathcal{F} = \mathcal{H}^\eta$ is a proof in S4$^\Delta$ [S4$'^\Delta$, S4$''^\Delta$]. Now let p be the proof assignment such that for any $\Box\phi^i$ in $\mathcal{F}$, $p(\phi, i) = \eta^{-1}(i)$. p will satisfy all the equations in $\mathbf{E}_\mathcal{F}$ in which the R2-equations are derived from $\mathcal{H}$, and then p is a characteristic proof assignment p on $\mathcal{F}$ and $\mathcal{F}^p$ is $\mathcal{H}$. ⊣

Definition 5.5.7. *Given a formula or a sequence of formulas $\mathcal{D}$ in $L_\Box$, we call a label function $r\colon m(\mathcal{D}) \to Tm$ a* realization *on $\mathcal{D}$.*

Each realization r induces a translation from $\mathcal{D}$ to a sequence of formulas $\mathcal{D}^r$ in $L_:$ such that for any $x \in m(\mathcal{D})$, $\mathcal{D}^r(x) = r(x){:}\mathcal{D}^r(x.\star)$.

Here is the proof realization result.

Theorem 5.5.8. *A proof $\mathcal{D}$ in S4, S4$'$, or S4$''$ is non-circular if and only if there exists a realization r such that $\mathcal{D}^r$ is a proof in GELP$^-$, ELP$^-$, or ELP, respectively.*

Proof. For the "only if" part, given a non-circular proof $\mathcal{D}$ in S4 [S4$'$, S4$''$], by Theorem 5.4.9, there is a numerical label function Δ on $\mathcal{D}$ such that $\mathcal{D}^\Delta$ is a proof in S4$^\Delta$ [S4$'^\Delta$, S4$''^\Delta$], and, by Theorem 5.5.6, there is a proof assignment p on $\mathcal{D}^\Delta$ such that $(\mathcal{D}^\Delta)^p$ is a proof in GELP$^-$ [ELP$^-$, ELP]. Let r be the realization on $\mathcal{D}$ such that for any $x \in m(\mathcal{D})$, $r(x) = p(\mathcal{D}^\Delta(x.\star), \Delta(x))$. Then $\mathcal{D}^r = (\mathcal{D}^\Delta)^p$, so r is what we are looking for. For the other direction, let $\mathcal{H} = \mathcal{D}^r$ be a proof in GELP$^-$ [ELP$^-$, ELP]. Then, by Theorem 5.5.6, $\mathcal{H} = \mathcal{F}^p$ for some proof $\mathcal{F}$ in S4$^\Delta$ [S4$'^\Delta$, S4$''^\Delta$] and some proof assignment p. It can be checked that $\mathcal{D} = \mathcal{F}^\Box$. So $\mathcal{D}$ is non-circular.

In fact, for this direction, we can directly prove it from $\mathcal{H}$. If we disregard the superficial symbolic difference, language $L_:$ is one of labeled modal languages $L_\mathbf{I}$ with proof terms as labels. Then $\mathcal{D} = \mathcal{H}^\Box$ and $l_\mathcal{H} = r$. However r will be a non-circular proof label function on $\mathcal{D}$ since it needs to satisfy the conditions of proof terms on the modal axiom schemes. ⊣

Definition 5.5.9. *Given a proof $\mathcal{D}$ in S4, S4$'$, or S4$''$, a realization r on $\mathcal{D}$ is called* characteristic *if there is an increasing label function Δ on $\mathcal{D}$ and a characteristic proof assignmentp on $\mathcal{D}^\Delta$ such that $\mathcal{D}^r = (\mathcal{D}^\Delta)^p$.*

Corollary 5.5.10. *$\mathcal{H}$=$\mathcal{D}^r$ is a proof in GELP$^-$, ELP$^-$, or ELP if and only if r is a characteristic realization on $\mathcal{D}$.*

The theorem realization result between S4-systems and LP-systems immediately follows.

Corollary 5.5.11. *An $L_\Box$ formula ϕ is a theorem of S4, S4′, or S4″ if and only if there is a realization r on ϕ such that ϕ^r is a theorem in GELP$^-$, ELP$^-$, or ELP, respectively.*

5.5.2 The Realization Theorem for LP

According to the algorithmic procedures we have so far, we are able to turn an S4G proof, to an S4$^\Delta$G proof, to an S4$^\Delta$ proof, to an S4$'^\Delta$ proof, to an ELP$^-$ proof, and hence an ELP proof, which is a supersystem of ELP$^-$. So given an S4 theorem ϕ, we are able to turn it into an ELP theorem ϕ^r. But if we want to realize an S4 theorem exactly to an LP theorem, we need one more step. One directly way is to find an algorithm converting an ELP proof to an LP proof by a proof term translation, but, to keep the flavor of this chapter, we will determine the subclass of S4$''^\Delta$ proofs, and hence the subclass of non-circular S4″ proofs, called *stable*, which is precisely the class of proofs can be realized to LP proofs.

We call a set of m-formulas *C-stable* for C in { A1, A2, A4, R2 }, if every element of the set is so. A set is *stable* if it is C-stable for some C.

Definition 5.5.12. *Let S be a set of m-formulas in an S4$'^\Delta$ proof $\mathcal{F}$. $P_*(S)$ is the set of all $*$-predecessors of m-formulas in S for $* \in \{\alpha, \beta, \gamma, \delta\}$ (see Definition 5.5.2). We say an equivalence relation $\sim$ defined on m-formulas in $\mathcal{F}$ is* stable *if it satisfies the following conditions:*

1. *every induced equivalence class with respect to $\sim$ is stable,*
2. *given an equivalence class E, for each $* \in \{\alpha, \beta, \gamma\}$, $P_*(E)$ is a subset of some equivalence class, and $P_\delta(E)$ is a subset of a union of at most two equivalence classes,*
3. *all the equivalence classes together form a linear finite chain E_1, E_2, $\ldots E_n$ such that for any $\phi \in E_i$, $\psi \in E_j$ with ϕ a predecessor of ψ, $i < j$.*

A proof is stable *if we can define a stable equivalence relation on the proof.*

Lemma 5.5.13. *Every stable S4$''^\Delta$ proof $\mathcal{F}$ has a simple characteristic proof assignment.*

Proof. Let E_1, E_2, $\dots E_n$ be a chain of equivalence classes of m-formulas in $\mathcal{F}$ such that if $\phi{\in}E_i$ is a predecessor of $\psi{\in}E_j$, then $i{<}j$. We will construct a simple characteristic proof assignment p such that for any m-formulas in the same equivalence class, the same proof term will be assigned. The construction is by induction on the index of the equivalence classes in the chain, and we will write $p(E_i){=}t$ to mean that the proof term t is assigned to all the m-formulas in E_i.

First of all, if there is an R2-formula in E_1, let $p(E_1){=}c$ for some proof constant c, otherwise $p(E_1){=}v$ for some proof variable v. Suppose that for any $i{<}k$, $p(E_i)$ is determined. If there is an A1-formula in E_k, then since the equivalence relation is stable, there are $i,j{<}k$ such that $P_\alpha(E_k){\subseteq}E_i$ and $P_\beta(E_k){\subseteq}E_j$. Let $p(E_k){=}p(E_i){\cdot}p(E_j)$. If there is an A2-formula in E_k, let $p(E_k){=}!p(E_i)$, provided $P_\gamma(E_k){\subseteq}E_i$. If there is an A4-formula in E_k, let $p(E_k){=}p(E_i){+}p(E_j)$, provided $P_\delta(E_k){\subseteq}E_i{\cup}E_j$ with $i{\leq}j$. If E_k is none of the above cases, then, following the procedure of dealing with E_1, assign a proof constant or proof variable to the m-formulas in E_k. One caveat here is that each time a new proof constant or a new proof variable is assigned. This step will help to create a *normal realization*, which will be discussed at the end of this section. Continue this process until all equivalence classes are assigned, then p is a simple characteristic proof assignment of $\mathcal{F}$. ⊣

Theorem 5.5.14. *An $S4''^{\Delta}$ proof $\mathcal{F}$ is stable if and only if there exists a proof assignment p such that $\mathcal{F}^p$ is a proof in LP.*

Proof. If p is a simple characteristic proof assignment, then it is easy to see that $\mathcal{F}^p$ is an LP proof, and, by the previous lemma, such a proof assignment exists for a stable proof, and hence the "only if" part of the theorem is proved. For the other part, we define an equivalence relation on the set of m-formulas in $\mathcal{F}$ such that $\Box\phi^i{\sim}\Box\psi^j$ if and only if $p(\phi,i){=}p(\psi,j)$. Put the induced equivalence classes in order, E_1, E_2, $\dots E_n$, such that $i{<}j$ if $p(E_i)$ is a subterm of $p(E_j)$ with $p(E)$ being the proof term assigned to all the m-formulas in E. Then each E_i is stable, since, if, say, there is an A1-formula in E_i, then all other m-formulas in E_i must not be an A2-, A4-, and R2-formula, because the application $\cdot$ is the main proof term operation of $p(E_i)$. Furthermore, $P_*(E_i){\subseteq}E_j$ for some $j{<}i$ with $*{\in}\{\,\alpha,\beta,\gamma\,\}$, and $P_\delta(E_i){\subseteq}E_j{\cup}E_k$ for some $j,k{<}i$. Therefore $\sim$, and hence $\mathcal{F}$, is stable. ⊣

Definition 5.5.15. *In an $S4''^{\Delta}$ proof, we call an A1-formula or A2-formula* standard*, if there is only one axiom in the whole proof in which the formula is the leading formula, and an A4-formula* standard *if there are at most two axioms in the whole proof in which the formula is the leading formula. An $S4''^{\Delta}$ proof is called* standard *if there is no non-standard formulas in the proof.*

Lemma 5.5.16. *Every standard $S4''^{\Delta}$ proof is stable.*

Proof. Basically, the identity relation between m-formulas is a stable equivalence relation. We can arrange the m-formulas in order based on their principal labels (the order of the formulas with the same principal labels does not matter), and then it can be checked that the identity relation satisfies all the conditions in Definition 5.5.12. ⊣

Here's the final step.

Proposition 5.5.17. *Every $S4''^{\Delta}$ proof can be extended to a stable proof.*

Proof. First of all, before extending the proof, if needed, the number labels will be modified. Notice that for any fixed numbers m, n, if we modify the number labels of the proof such that for any $k>m$, n is added to k, then the result is still an $\mathrm{S4}''^{\Delta}$ proof (all the modal axioms are still modal axioms). Hence we can always modify the proof such that the difference between two consecutive number labels is as wide as we would like it to be.

Let $\Box\phi^i$ be a non-standard m-formula in an $\mathrm{S4}''^{\Delta}$ proof, and $\{\psi_j\}_{0\leq j\leq n}$ be the class of axioms in which $\Box\phi^i$ is the leading formula. We will add several formulas, including several axioms, into the proof such that for each ψ_j, it is no longer an axiom but a derived formula in the proof, and, although new axioms are included, no additional non-standard m-formulas are added to the proof. Therefore, after this procedure, the overall number of non-standard m-formulas in the proof is decreased. Continuing this process, a standard proof will be built.

The formulas we add to the proof are the following. First, we will add axioms ψ'_j into the proof, where ψ'_j is the axiom ψ_j with m-formula $\Box\phi^{k+2j}$ substituting for $\Box\phi^i$ as the leading formula of the axiom. The number k has to be carefully chosen such that $k+2n<i$, and for each j, $\Box\phi^{k+2j}$ is new to the proof and ψ'_j is still an axiom. This is the stage of the procedure where modification of the proof might be needed. We also add axioms $\Box\phi^k\to\Box\phi^{k+1}$, $\Box\phi^{k+2n-1}\to\Box\phi^i$, $\Box\phi^{k+2n}\to\Box\phi^i$, and $\Box\phi^{k+2j-1}\to\Box\phi^{k+2j+1}$ and $\Box\phi^{k+2j}\to\Box\phi^{k+2j+1}$ for $1\leq j\leq(n-1)$. Then, we add more formulas such that ψ_j are derived from these axioms. Furthermore, some of the axioms ψ_j might be applied by the rule of axiom necessitation in the original proof, but now they are not axioms. The last step is to add more formulas with the method discussed in Proposition 5.4.18 or 5.4.19 such that $\Box\psi_j$ is also derived in the extended proof. This completes the procedure. ⊣

Theorem 5.5.18 (Realization Theorem).
A formula $\phi\in L_\Box$ is an S4 theorem if and only if there is a realization r such that ϕ^r is an LP theorem.

In Artemov (2001), a special type of realization, called *normal realization*, is highlighted. It requires that the negative modal occurrences of an S4 theorem be realized to proof variables. From the procedure we introduce here, it should be clear that only initial modalities in a proof can be realized to variables. However, this is just the case, which is witnessed by the fact that all the negative m-formulas of an S4 theorem can be assigned with the numerical label 0 when a cut-free Gentzen style proof of the theorem is translated to an $S4^{\Delta}$ proof. Observing the Gentzen style proof, we can actually, if we want, assign different numerical labels to different negative modal occurrences to form different labeled m-formulas, and these labeled m-formulas will be kept to be initial in the derived standard $S4''^{\Delta}$ proof and hence realized to proof variables.

5.6 Discussions

The two main topics of this chapter are the completeness of non-circular S4 proofs, and the proof realization procedure from S4 to LP. The logical framework of $S4^{\Delta}$ is introduced to bridge the topics. Accordingly, our work establishes a structural property of Hilbert style proofs, a subject hardly practiced in the literature yet. For LP, as we know, the proof terms can be interpretated as justification entities or explicit proofs in a formal arithmetic system. It, as well as $S4^{\Delta}$, with numerical labels interpreted as reasoning time or proof lengths, has its own area of applications and hence a logical framework worth further investigation for its own sake. But, as far as the non-circular proof is concerned, they also serve well as technical tools for the study of the structure of S4 proofs, with proof terms or numerical labels interally recording the relations between subformulas within a proof.

The discussions in this work can be well adapted to modal logics with cut-free Gentzen proof systems without much difficulty. Given a class of non-circular proofs with respect to a stamp of a modal logic, the procedure realizing these proofs into proofs in the explicit counterpart of the modal logic is not difficult to figure out. Even if variants of the counterpart are concerned — with necessitation or axiom necessitation; ELP-like or LP-like, we have provided satisfactory algorithms to handle these cases. The only step that we need to especially take care of is a proof of the class of non-circular proof is complete; and this is equivalent to, according to the method we provide here, search for suitable Δ-counterparts of the modal rules in the cut-free Gentzen system associated to the modal logic such that these rules are derivable in the Δ-version of the modal logic from which the class of non-circular proof in discussion can be defined. At the end of this chapter, we discuss the cases of modal logics S5 and GL as two examples.

The system of S5$^{\Delta}$ that we are going to discuss is S4$^{\Delta}$ with the addition of the following modal axiom: "$\neg\Box F^i \rightarrow \Box(\neg\Box F^i)^j$, $j>i$." Although there is still no satisfying elegant cut-free Gentzen style proof system developed yet, Fitting provided one in Fitting (1999b) with the catch that in order to prove ϕ, it is a proof tree of $\Rightarrow\Box\phi$, instead of $\Rightarrow\phi$, to be constructed.[4] But this catch doesn't affect the applicability of our method. Let S5$^{\Delta}$G be the system of S4$^{\Delta}$G introduced above with the following R$\Box$ rule instead:

$$\frac{\Box\Gamma^{\iota} \Rightarrow \Box\Gamma'^{\iota'}, A}{\Box\Gamma^{\iota} \Rightarrow \Box\Gamma'^{\iota'}, \Box A^{k}},$$

for any k, if both ι and ι' are empty, and for any $k > \max(\iota, \iota') + (|\Gamma| + |\Gamma'|)$, otherwise. It can be proved that this rule and, of course, the L$\Box$ rule are derivable in S5$^{\Delta}$. Finally, it follows that if ϕ is an S5 theorem, there is an S5$^{\Delta}$ proof of $\Box\phi^i$ for some i, and hence there is a non-circular proof of $\Box\phi$. Applying the reflexive axiom A3, $\Box\phi \rightarrow \phi$, and modus ponens, we produce a non-circular proof of ϕ.

The Δ-versions of GL we are going to discuss are S4$^{\Delta}$ with the axiom scheme A3 of S4$^{\Delta}$ replaced by the axiom scheme "$\Box(\Box F^j \rightarrow F)^i \rightarrow \Box F^k$, $k > f(i,j)$," with $f(i,j) = i$, j, or $\max(i,j)$, respectively. That is, we consider three versions of GL$^{\Delta}$ at once. Their associated cut-free Gentzen systems are S4$^{\Delta}$G without the L$\Box$ rule and with the following R$\Box$ rule:

$$\frac{\Box\Gamma^{\iota}, \Gamma, \Box A^{j} \Longrightarrow A}{\Box\Gamma^{\iota} \Longrightarrow \Box A^{k}},$$

for any $k > f(0,j)$, if $|\Gamma| = 0$ and for any $k > f(k', j)$ with $k' = \max(\iota) + 2|\Gamma|$, otherwise, and again it can be checked that these rules are derivable in their corresponding systems of GL$^{\Delta}$.

[4]The system introduced in Fitting (1999b) is actually a Tableaux system. But the corresponding Gentzen system is not difficult to derive. Our S5$^{\Delta}$G introduced here is similar to the corresponding Gentzen system but with the language with only one modality and with the additional considerations of numerical labels.

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