Interpreting the plus sign in justification logic as the union of sets of justifications

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Abstract

In the study of justification logic, the role played by the proof term operator + always needs further explication. In this paper we will clarify its function in the project, and propose a variant justification logical system in which + gets the intuitive interpretation of set-theoretical *union* of proof terms. The intended epistemic semantics for the system will be given, followed by the completeness theorem. We will then investigate the formal relation between the variant and the original logical system of justification, and this investigation leads to the *realization theorem* for the new system. Finally, we will discuss the possible arithmetic semantics that can be supplemented to the proposed system.

1 Introduction

The research project of designing logical devices for reasoning about knowledge and belief has been widely conducted these days, and justification logic is a welcome addition. With the enhanced expressivity of justification logic, not only the logical relations of agent's knowledge or belief can be reasoned, but also the reasoning procedures through which agent's new knowledge or belief is gained can be formulated. The framework of justification logic that is popularly studied these days is axiomatically introduced in [3], with the possible-world-like semantics in [6]. One important feature of the framework is that it bears a close relation to the traditional modal approach of epistemic logic. With the connection, it is then possible to search for the new results in the framework of justification logic in light of what have been established in the study of modal epistemic logic.[1]

Historically, the first justification logic is introduced through a reinterpretation of *Logic of Proofs*, abbreviated as LP, a system introduced in a project of providing formal BHK-semantics to intuitionistic logic. The process of the project is about the following: first, we embed intuitionistic logic, as suggested by Gödel, into the modal logic S4 with the modal operator read as provability, and then embed, as put forward by Artemov [3], S4 into LP, which is understood as a logical system of formal proofs in formal arithmetic. The latter connection is established through a formal theorem called the *realization theorem.* The theorem, roughly speaking, states that every S4 theorem can be turned into a theorem in LP by substituting for the provability modalities suitable *proof terms*, which are the structured objects in LP employed to represent the formal proofs. This formal connection is also important in the context of epistemic reasoning. While S4 has long been regarded as logic of knowledge [9], the realization theorem tells

[1]For more information of justification logic, see [2].

us that there are logical relations of justification underneath the theorems in S4, which only can be revealed in the formalism of Logic of Proofs with proofs terms read as justifications.

The proof term operators that combine proof terms in the formalism of LP are $\cdot$, !, and $+$, where the binary $\cdot$ is intended to denote the application of *modus ponens*, and the unary ! is to denote the proof checker or the operation of positive introspection in the reading of justification logic. The need and the interpretation of these two operators are not difficult to understand, and it is the operations on proofs denoted by $\cdot$ and ! that are discussed in [8] by Gödel.[2] With only these two operators, however, not all S4 theorem can be realized. Consider the S4 theorems of the following kind:

(1) $\Box A \vee \Box B \to \Box(A \vee B)$.

Given that a and b are the proof terms realizing the provability modalities of A and B respectively, some kind of combination of a and b is certainly in need to realize the provability of $A \vee B$, and hence $+$ is introduced with the associated axioms to address the realization problem.[3] The main function of $+$ is that given any proof terms s and t, $s + t$ should be able to prove whatever can be proved by s and t. Now given the function, and suppose that $+$ denotes an operation on proofs, which means $s + t$ is a genuine proof, then it immediately follows that the proofs represented by proof terms, including terms of the form $s+t$, are *multi-conclusion proofs*. Or consider an intuitive concrete interpretation of $+$ as a genuine operation that $s+t$ is the proof built up by the *concatenation* of proofs s and t, and then every formulas in the proof is regarded as a proved formula of the proof.[4] This generalized concept of proofs is not an obscure or unworkable concept, but just not the one that we normally use in our discourse of discussing formal proofs. This uncommon generalization complicates our understanding and appreciation of the system, and it should not be essential to the success of a project on the formal arithmetic semantics of intuitionistic logic.

In the epistemic context, we don't have a good interpretation of $+$ either. Given justifications s and t, a straightforward interpretation of $s + t$ would be the *combination* of justifications s and t.[5] But what is the combination? Since from the study of common-sensical reasoning, we are learning that not all the acting of justification is monotonic.[6] That is, more justification sometimes will even cripple our confidence in what have been taken as justified. However, as the role $+$ plays in the proposed systems of justification logic, some kind of

[2]To be more precise, it is the application of *cut rule*, not *modus pones*, discussed in [8].

[3]The proof term for the realization of the provability of $A \vee B$ in the case of (1) is normally $(c\cdot a)+(d\cdot b)$ for suitable constants c, d, where $c\cdot a$ proves $A \vee B$ if a proves A, and $d\cdot b$ proves $A \vee B$ if b proves B. Since either a proves A or b proves B, so either $c\cdot a$ proves $A \vee B$ or $d\cdot b$ proves $A \vee B$; now with the function of $+$, $(c\cdot a)+(d\cdot b)$ proves $A \vee B$. See the system of LP in the next section for an idea of the formal proof.

[4]Cf.[2] in which such a reading of $+$ as *concatenation* is suggested. Nonetheless, although this interpretation is quite intuitive, helping us quickly have an idea of the function of $+$, it only could be an informal, not a formal, interpretation of $+$ in LP, since under this interpretation a formula is provable by $s + t$ if and only if it is provable by $t+s$, and a theorem of this kind can't be obtained in LP. See more discussion on this point in Section 6.

[5]See [4].

[6]See the entries of *non-monotonic logic* and *defeasible reasoning* in Stanford Encyclopedia of Philosophy [16, 10].

monotonicity seems built into the function of $+$. So without more explicit explanation of what the combination is, this interpretation gives us more difficulty rather than reduces it in understanding the systems and seeking applications. Notice that the question raised here is only when $+$ is treated as an operation on justifications, and hence an interaction between justifications is assumed and an interpretation to the interaction is needed. But once we are free from the thought that the $s + t$ denotes a single justification, then we escape from the problem.

There is, however, an intuitive interpretation of $+$ as it has the main task in the realization of S4 theorems. That is to take it to function like the set-theoretical operation *union*. So $s + t$ is simply the set of justifications s and t, without assuming any interaction between these justifications, and the formulas which are considered to be proved or justified by this set is simply the collection of formulas to be proved or justified by s and t. This simple interpretation can't be obtained for the systems of justification logic that are popularly studied these days; they are not introduced according to this interpretation. The main goal of this paper is thus to improve the project of justification logic by introducing a system which can be equipped with such an intuitive interpretation to the proof term operator $+$.

At the end of the paper, we will sketch the possible arithmetic interpretations of our system in which $+$ will be interpreted as *union of proofs*, the interpretation that we prefer, and compare the system with its interpretations to some other works where arithmetic considerations has been discussed, including the system in [1], whose axiomatization has a similar look to our system.

For the simplification of the discussion, in this paper we will only consider the justification logical system with respect to the modal epistemic logic S4; that is, we are concerning with a variant system of LP. Many other justification logics corresponding to the super- or sub-logics of S4 have been proposed in the literature; however, it will not be difficult to see that our investigation on the function of $+$ can be easily adapted to these logics. In the next section we will present the system of LP for the purpose of comparison. The system of the current project, call it LP^*, will be introduced in Section 3, with its intended epistemic semantics and the completeness theorem. Some properties of the system will be provided in Section 4. Section 5 is the realization theorem for LP^*. The arithmetic discussion of LP^* will be conducted in the final section.

2 LP

The language of LP is introduced by two tires. The first is the proof terms or justification terms, which are built up from justification constants and variables by unary operator !, and binary $\cdot$ and $+$. Then the well-formed formulas are built up from a set of sentential letters and truth-functional connectives in the usual way with an additional formation rule: if t is a justification term and A is a well-formed formula, $t{:}A$ is also a well-formed formula. The first axiom system of justification logic, also known as Logic of Proofs, LP, is the following:

Axioms:

A0 complete finite axiom schemes of classical propositional logic
A1 $s{:}(A \to B) \to (t{:}A \to s{\cdot}t{:}B)$

A2 $s{:}A \to\, !s{:}s{:}A$
A3 $s{:}A \to A$
A4 $s{:}A \to (s+t){:}A$
A5 $t{:}A \to (s+t){:}A$

Rules:

R1 $\dfrac{A \to B \quad A}{B}$ (Modus Ponens)

R2 $\overline{c : A}$, for c a constant, and A an above axiom (axiom necessitation)

A *Constant Specification*, $\mathcal{CS}$ is a set of formulas of the form $c{:}A$ with c a constant, and A an above axiom. Given a constant specification $\mathcal{CS}$, $\mathsf{LP}(\mathcal{CS})$ is a subsystem of LP where only R2 with consequences in $\mathcal{CS}$ are allowed to apply. So LP itself is a special case of $\mathsf{LP}(\mathcal{CS})$, with $\mathcal{CS}$, called *total*, the maximal constant specification. Initially, $\mathsf{LP}(\mathcal{CS})$ is introduced for the purpose of arithmetic completeness, but in the context of justification logic its introduction gives the great flexibility to model an agent with restricted reasoning ability. That is, simply put, an $\mathsf{LP}(\mathcal{CS})$ agent can only use the formula A with $c : A$ in $\mathcal{CS}$ to reason about, and c is the justification that the agent employs to justify the formula A.

Several semantics have been proposed for LP ($\mathsf{LP}(\mathcal{CS})$), with the first, the arithmetic semantics, given in [3]. The wildly studied epistemic semantics is introduced in [6], which is an augmented possible world semantics with each world equipped with a machinery called *evidence function* to record the modeled agent's reasoning results and their corresponding justifications at the world. The semantics introduced in [15] turns out to be the semantics of one world model of the epistemic semantics, which is useful in determining the decidability and complexity of justification logics [13]. The interested readers refer to the aforementioned papers and [2] for details.

Some remarks are in order. First, it is easy to see that there is some parallelism between this presentation and the standard axiom system of $\mathsf{S4}$. But normally in $\mathsf{S4}$ we have the *necessitation rule*, i.e. $\Box A$ is deducible from A, instead of the axiom necessitation. However, (1) the axiom system where the axiom necessitation (restricting the application of *necessitation* to axioms) takes place of the necessitation rule in the standard $\mathsf{S4}$ axiom system is still a system of $\mathsf{S4}$; (2) a rule similar to the necessitation rule is derivable in LP. It has been shown, in the name of *internalization*, that if A is provable in LP, then there is a closed term t such that $t{:}A$ is provable in LP, where a closed term is a term without variables. This import of the axiom necessitation into the system is important either for the project in the formal arithmetic discourse or in the epistemic context. With the axiom necessitation, we can fully trace the procedures of reasoning through the analysis of proof terms.

Second, the system is presented as usual in a schematic way, but, unusually, schematic in two categories of formal objects. While A, B are metavariables ranging over all well-formed formulas, s, t ranging over proof terms.

Third, except A4 and A5, the axioms in the axiom system above are corresponding to axioms in the standard $\mathsf{S4}$ system. This can be understood as that each of these axioms in LP are revealing the complicated proof or justification structures hidden in the corresponding $\mathsf{S4}$ axioms, where the simple, unstructured modal operator, box, is in use. This is also a point showing that from the beginning the introduction of $+$ for the combination of proof terms has a different origin from that of the other term operators $\cdot$ and !. So it makes sense to have it treated differently. But this is not the approach taken by LP. As the way that the axioms A4 and A5 are introduced, $+$ is treated, on a par with the other term operators, as representing a genuine operation on proofs or justifications, an operation of which, as we have discussed, a good candidate has hardly been found. We can also immediately see that the set-theoretical interpretation of $+$ in LP as the union operator is not the way to go. Though from A4 and A5, we can easily derive $s{:}A \vee t{:}A \rightarrow (s+t){:}A$, neither the reverse of the conditional, nor $(s+t){:}A \leftrightarrow (t+s){:}A$, can be derived in LP. However, these non-theorems in LP are apparently the characteristics that would be enjoyed by the union reading of $+$.

There is one consideration that will invite us to treat $+$ no differently from the other term operators. The schematic presentation of the axiom schemes demands that all terms, including the term build up from $+$, satisfy the axioms from A1 to A3. In the literature, we can find an LP variant, system of SLP ([1]), introduced in this way and the system has some similarity to our system. SLP is originated from arithmetic consideration, but supplied with an epistemic-like semantics.[7] The system is extended from LP by having additional axiom schemes $u{\cdot}(s{+}t){:}A \leftrightarrow u{\cdot}s{:}A \vee u{\cdot}t{:}A$ and $(s{+}t){\cdot}u{:}A \leftrightarrow s{\cdot}u{:}A \vee t{\cdot}u{:}A$, which includes, by notational convention, $(s{+}t){:}A \leftrightarrow s{:}A \vee t{:}A$, where s, t, u are metavariables ranging over proof terms, and the semantics of SLP is augmented from the one-world model semantics of LP by, corresponding to the two additional axioms, having two more bi-conditional conditions on the *evidence function* concerning the interaction between $\cdot$ and $+$. From the way the axiom system is presented, and the way the semantics is proposed: the behavior of $+$ is governed by the *evidence function*, $+$ in SLP is treated as an operation on the proofs or justifications, and, since the plus sign is treated in this way, the need of the axiom schemes and semantic conditions in SLP concerning the relation between $+$ and ! becomes urgent. Nonetheless, our goal is to treat $+$ as the union of proof terms, then there is no apparent reason to demand the terms containing $+$ also satisfy conditions determined by the axioms A1 to A3. And it is this consideration that will be taken care of in the following introduction of LP*.

3 LP*

The language of LP* is similar to the language of LP, but the terms are introduced in two tiers. The *proof terms*, or *justification terms*, are built up from proof constants and variables, and operators $\cdot$ and !. Then the *super proof terms*, or just called *super terms*, are built up from proof terms and $+$. The well-formed formulas of LP* are similarly built up from a set of sentential letters and truth-functional connectives but with the additional formation rule: if p is a super proof term and A is a well-formed formula, $p{:}A$ is also a well-formed formula. The system we are proposing is as follows:

Axioms:

A0 complete finite axiom schemes of classical propositional logic
A1 $s{:}(A \rightarrow B) \rightarrow (t{:}A \rightarrow s{\cdot}t{:}B)$
A2 $s{:}A \rightarrow\ !s{:}s{:}A$

[7] More discussion on the arithmetic consideration of SLP is in Section 6

A3 $s{:}A \to A$

A4 $p{:}A \vee q{:}A \leftrightarrow (p+q){:}A$

Rules:

R1 $\dfrac{A \to B \quad A}{B}$ (Modus Ponens)

R2 $\overline{c : A}$, for c a constant, and A an above axiom (axiom necessitation)

The *Constant Specification*, $\mathcal{CS}$, for LP^* and $\mathsf{LP}^*(\mathcal{CS})$ are defined in the same way as those for LP but on the language of LP^*.

So here we present a system schematically in three categories: A, B are metavariables for well-formed formulas, s, t for proof terms, and p, q for super terms. As we can immediately see, in LP^* $+$ is some kind of commutative, and super terms are not demanded to satisfy the conditions set by the axioms A1 to A3.

Before presenting the intended epistemic semantics, we would like to propose an adjustment on the semantics of justification logic that we normally encounter. The ideology of the popular possible world semantics for justification logic is that in each world both the static basic facts of the world and the process of agent's reasoning in the world will be taken down, where the former is recorded by the truth value of sentential letters and the latter by a devise of *evidence function*, which reflects the reasoning ability we assume the agent to possess. Here we suggest a direct way to model the agent's reasoning ability, and the justification procedure taken by the agent. We suggest to employ a proof system to model the way that the agent uses to generate new knowledge or belief. Given a constant specification $\mathcal{CS}$ in LP^*, we call a proof system of the language LP^* *Internal* $\mathsf{S4}$ *with respect to* $\mathcal{CS}$, abbreviated as $\mathsf{IS4}(\mathcal{CS})$, if its axioms are formulas $c{:}A$ in $\mathcal{CS}$, and the rules are the next two:

$$\frac{s{:}(A \to B) \qquad t{:}A}{s{\cdot}t{:}B} \qquad\qquad \frac{s{:}A}{!s{:}s{:}A} .$$

By $\mathsf{IS4}$, we mean $\mathsf{IS4}(\mathcal{CS})$ with $\mathcal{CS}$ total. We will write $\Gamma \vdash_{\mathsf{IS4}(\mathcal{CS})} A$ to denote that there is a derivation from the set Γ of well-formed formulas to A in the system of $\mathsf{IS4}(\mathcal{CS})$. The system itself is not new. It is first introduced in [11] for the study of the so-called *reflected fragment of* LP, where a rule introducing term $s{+}t$ is also included.[8] But to my knowledge, no suggestion to put it in an epistemic model has been proposed yet, and this proposal, as we can see lately, also simplifies the completeness proof.

By a $\mathsf{LP}^*(\mathcal{CS})$ model, we means a quadruple $\langle W, R, \pi, \upsilon\rangle$, where W is a set of possible worlds, and R is a reflexive and transitive binary relation on W, i.e., $\langle M, R\rangle$ an $\mathsf{S4}$ frame, π is a function from the worlds to sets of sentential letters, and υ is a function from the worlds to sets of formulas of the form $t{:}A$, for proof term t and well-formed formula A with the condition $\upsilon(w) \subseteq \upsilon(w')$ for wRw'.

Given an $\mathsf{LP}^*(\mathcal{CS})$ model M and a world w in the model, the truth conditions for the truth-functional connectives are as usual, and for a proof term t and super terms p, q we have

$M, w \vDash t{:}A$ if and only if $\upsilon(w) \vdash_{\mathsf{IS4}(\mathcal{CS})} t{:}A$ and for all worlds wRw', $M, w' \vDash A$;
$M, w \vDash (p+q){:}A$ if and only if $M, w \vDash p{:}A$ or $M, w \vDash q{:}A$.

[8] It is called $\mathsf{C}_{\mathcal{CS}}$ in [11] with the rule concerning $+$. A similar calculus also can be found in [13], called $*!_{\mathcal{CS}}$-calculus therein.

Here $v(w)$ collects all the formulas $s{:}B$ where B is the formula the agent holds to be true for a primitive justification s in the world w, and $t{:}A$ is true in the world of w if and only if A is true in all possible worlds relative to w, and the agent is able to derive A by t from his primitive understanding of the world.

From the semantics we can easily have the following consequence. Let $(\cdot)^*$ be the function assigning sets of proof terms to super terms such that $(t)^* = \{t\}$ and $(p+q)^* = p^* \cup q^*$.

Lemma 3.1. *For a super term p, $M, w \vDash p{:}A$ if and only if there is a proof term $t \in p^*$, $M, w \vDash t{:}A$.*

This is a desirable consequence, showing that $+$ in LP^* functions like the union of sets. Sometimes we will write $\sum_{s_i \in p^*} s_i$ or just $\sum s_i$ for p to explicitly indicate the summands of p. Both $\cdot$ and ! have the binding force stronger than Σ.

To show the completeness result, we use the standard canonical model method. We call a set of LP^* well-formed formulas Γ, $\mathsf{LP}^*(\mathcal{CS})$-inconsistent if $\neg(A_1 \wedge \ldots \wedge A_n)$ is $\mathsf{LP}^*(\mathcal{CS})$ provable for a finite number of formulas $A_i \in \Gamma$; otherwise, $\mathsf{LP}^*(\mathcal{CS})$-consistent. And to create a maximally consistent set from a consistent set, the Lindenbaum theorem works well for $\mathsf{LP}^*(\mathcal{CS})$. $\Gamma^\# = \{A | t{:}A \in \Gamma\}$. For an LP^* formula A, $\vdash_{\mathsf{LP}^*(\mathcal{CS})} A$ means that A is provable in $\mathsf{LP}^*(\mathcal{CS})$, and $\vDash_{\mathsf{LP}^*(\mathcal{CS})} A$ that A is true in all worlds of all $\mathsf{LP}^*(\mathcal{CS})$ models, i.e., A is $\mathsf{LP}^*(\mathcal{CS})$ valid.

Theorem 3.2. *$\vdash_{\mathsf{LP}^*(\mathcal{CS})} A$ if and only if $\vDash_{\mathsf{LP}^*(\mathcal{CS})} A$.*

Proof. The proof of the soundness part is straightforward. I will show some cases for demonstration. For a model $M = \langle W, R, \pi, v\rangle$ and a world $w \in W$, if $M, w \vDash t{:}A$ then, since R is transitive, for all w' and w'' with wRw' and $w'Rw''$, $M, w'' \vDash A$, and, since $v(w) \subseteq v(w')$ and $v(w) \vdash_{\mathsf{IS4}(\mathcal{CS})} t{:}A$, $v(w') \vdash_{\mathsf{IS4}(\mathcal{CS})} t{:}A$. So $M, w' \vDash t{:}A$. Moreover, $v(w) \vdash_{\mathsf{IS4}(\mathcal{CS})} !t{:}t{:}A$, so $M, w \vDash !t{:}t{:}A$. This justifies the axiom A2. Now suppose we have shown that all axioms are true in all worlds w of all models M. Since for $c{:}A \in \mathcal{CS}$ with A an axiom, $v(w) \vdash_{\mathsf{IS4}(\mathcal{CS})} c{:}A$, $M, w \vDash c{:}A$. This justifies the rule of R2.

Now let W be the set of $\mathsf{LP}^*(\mathcal{CS})$ maximally consistent sets, and R the binary relation on W such that $\Gamma R \Gamma'$ if and only if $\Gamma^\# \subseteq \Gamma'$. Also let π and v be functions on W with $\pi(\Gamma)$ the set of sentential letters in Γ and $v(\Gamma)$ the set of all formulas of the form $t{:}A$ in Γ for proof terms t and well-formed formulas A. Then it can be shown that $\langle W, R\rangle$ is an $\mathsf{S4}$ frame and for $\Gamma R \Gamma'$, $v(\Gamma) \subseteq v(\Gamma')$, since if $t{:}A \in v(\Gamma)$, then $t{:}A \in \Gamma$, $!t{:}t{:}A \in \Gamma$, $t{:}A \in \Gamma^\#$, $t{:}A \in \Gamma'$, and $t{:}A \in v(\Gamma')$. So $M = \langle W, R, \pi, v\rangle$ is an $\mathsf{LP}^*(\mathcal{CS})$ model, call it the canonical model. Notice that for any $\Gamma \in W$, $\Gamma \vdash_{\mathsf{LP}^*(\mathcal{CS})} A$ if and only if $A \in \Gamma$, and it is not difficult to show that $v(\Gamma) \vdash_{\mathsf{IS4}(\mathcal{CS})} t{:}A$ if and only if $t{:}A \in v(\Gamma)$.

For the Truth Lemma, we skip the cases for truth-functional connectives. If $M, \Gamma \vDash t{:}A$, then $v(\Gamma) \vdash_{\mathsf{LP}^*(\mathcal{CS})} t{:}A$. So $t{:}A \in v(\Gamma)$, and $t{:}A \in \Gamma$. And if $t{:}A \in \Gamma$, then $t{:}A \in v(\Gamma)$ and $A \in \Gamma^\#$. So $A \in \Gamma'$ for all $\Gamma R \Gamma'$. Then $M, \Gamma' \vDash A$ by induction hypothesis and $M, \Gamma \vDash t{:}A$. Furthermore, $M, \Gamma \vDash (p+q){:}A$ if and only if $M, \Gamma \vDash p{:}A$ or $M, \Gamma \vDash q{:}A$ if and only if $p{:}A \in \Gamma$ or $q{:}A \in \Gamma$ if and only if $p{:}A \vee q{:}A \in \Gamma$ if and only if $(p+q){:}A \in \Gamma$. This completes the proof of Truth Lemma.

Finally, if A is not $\mathsf{LP}^*(\mathcal{CS})$ provable , $\{\neg A\}$ is $\mathsf{LP}^*(\mathcal{CS})$-consistent, which can be extended to a maximally consistent set Γ. By Truth Lemma, $M, \Gamma \vDash \neg A$, and hence $M, \Gamma \nvDash A$ for the canonical model M. This completes the proof. □

Remind that LP^* is $\mathsf{LP}^*(\mathcal{CS})$ with $\mathcal{CS}$ to be total, the maximal constant specification.

4 Some Properties

In this and the next section, we discuss some properties of LP^*. It is worth mentioning that all these properties are also held by $\mathsf{LP}^*(\mathcal{CS})$ with $\mathcal{CS}$ *axiom appropriate and schematic*, where $\mathcal{CS}$ is *axiom appropriate* if every LP^* axiom A is accompanied with a proof constant c such that $c{:}A \in \mathcal{CS}$, and *schematic* if every axiom instance A of the same axiom scheme is accompanied with the same constant c such that $c{:}A \in \mathcal{CS}$.

The first thing is that LP^* also enjoys the internalization property. That is, if A is LP^* provable or valid, so is $t{:}A$, for some term t and with some syntactic argument, we can restrict t to a closed proof term. Some connections between LP^* provability and $\mathsf{IS4}$ provability can be built. For any proof term t, and LP^* formula A, $\vdash_{\mathsf{LP}^*} t{:}A$ if and only if $\vdash_{\mathsf{IS4}} t{:}A$, and for any super term p, $\vdash_{\mathsf{LP}^*} p{:}A$ if and only if $\vdash_{\mathsf{IS4}} t{:}A$ for some proof term t.

For a formula A, we will write $A(x_1, \ldots, x_n)$ to indicate the variables of our focus in the formula, where $x_1, \ldots, x_n$ need not all be the variables in A.We will also write $A(t_1, \ldots, t_n)$ for terms $t_1, \ldots, t_n$, to denote the result of substituting terms t_i for variables x_i in A, which will be clear in the context. With the schematic presentation of the axiom system, we can see that if $A(x_1, \ldots, x_n)$ is an axiom in LP^*, so are $A(t_1, \ldots, t_n)$, and, established by induction on the length of the proof, if $A(x_1, \ldots, x_n)$ is provable so are $A(t_1, \ldots, t_n)$. Furthermore, there is a fixed closed proof term t such that $t{:}A(t_1, \ldots, t_n)$ is provable for any $t_1, \ldots, t_n$, if $A(x_1, \ldots, x_n)$ is provable.

As we have mentioned, the super terms need not to satisfy the conditions set by the axioms from A1 to A3, and hence certainly behave differently from the proof terms in LP. However, some similar conditions on these terms can be established, and this is the subject of the ensuing discussions of this section. Since for proof terms s, t, u in LP^*, $s{:}(A \to B) \to (t{:}A \to s{\cdot}t{:}B)$ and $s{:}(A \to B) \to (u{:}A \to s{\cdot}u{:}B)$ are both LP^* theorems, then we can derive, by applying propositional logic, that $s{:}(A \to B) \to (t{:}A \vee u{:}A \to s{\cdot}t{:}B \vee s{\cdot}u{:}B)$, and hence, by applying Axiom A4, that $s{:}(A \to B) \to ((t{+}u){:}A \to (s{\cdot}t{+}s{\cdot}u){:}B)$. This gives us a hint of the following lemma:

Lemma 4.1. $\vdash_{\mathsf{LP}^*} \sum s_i{:}(A \to B) \to (\sum t_j{:}A \to \sum s_i{\cdot}t_j{:}B)$.

Proof. We use semantic argument to show this. Given a model $M = \langle W, R, \pi, \upsilon \rangle$ and a world $w \in W$, suppose $M, w \vDash \sum s_i{:}(A{\to}B)$, and $M, w \vDash \sum t_j{:}A$, then there is an s_i and a t_j such that $M, w \vDash s_i{:}(A{\to}B)$ and $M, w \vDash t_j{:}A$. It follows that $M, w' \vDash A{\to}B$ and $M, w' \vDash A$ for all wRw', and therefore $M, w' \vDash B$, and furthermore that $\upsilon(w) \vdash_{\mathsf{IS4}} s_i{:}(A{\to}B)$ and $\upsilon(w) \vdash_{\mathsf{IS4}} t_j{:}A$, and therefore $\upsilon(w) \vdash_{\mathsf{IS4}} s_i{\cdot}t_j{:}B$. So $M, w \vDash s_i{\cdot}t_j{:}B$, and hence $M, w \vDash \sum s_i{\cdot}t_j{:}B$. □

The case of ! is more complicated. For any proof term s and super term $p = \sum s_i$ where s is one of p's summand, $s{:}A \to p{:}A$ is provable by applying classical propositional logic and the axiom A4, and hence there is a closed proof term e_s^p such that $e_s^p : (s{:}A \to p{:}A)$ is provable. To simplify the notation, we will write e to denote these terms. Since $s{:}A \to !s{:}s{:}A$ and $e{:}(s{:}A \to p{:}A) \to (!s{:}s{:}A \to e{\cdot}!s{:}p{:}A)$ are both LP^* provable, so are $s{:}A \to e{\cdot}!s{:}\sum s_i{:}A$ and $s{:}A \to \sum e{\cdot}!s_i{:}\sum s_i{:}A$. We have the following lemma:

Lemma 4.2. $\vdash_{\mathsf{LP}^*} \sum s_i{:}A \to \sum e{\cdot}!s_i{:}\sum s_i{:}A.$

Since $\vdash_{\mathsf{LP}^*} \sum s_i{:}A \to \bigvee s_i{:}A$ and $\vdash_{\mathsf{LP}^*} s_i{:}A \to A$, we have

Lemma 4.3. $\vdash_{\mathsf{LP}^*} \sum s_i{:}A \to A.$

And obviously, the following:

Lemma 4.4. $\vdash_{\mathsf{LP}^*} p{:}A \to (p+q){:}A$ *and* $\vdash_{\mathsf{LP}^*} q{:}A \to (p+q){:}A.$

Now we introduce symbols $\textcircled{!}$ and $\odot$ to make the following abbreviations: for super terms $p = \sum s_i$, and $q = \sum t_j$, $p \odot q = \sum s_i{\cdot}t_j$, and $\textcircled{!}p = \sum e{\cdot}!s_i$. To simplify the discussion, we assume that the same set of constants and variables and the same set of sentential letters are used by LP and LP^*. Let $(\cdot)^\circ$ be the translation from LP proof terms to LP^* supper terms such that constants and variables are fixed,and $(s \cdot t)^\circ = s^\circ \odot t^\circ$, $(!s)^\circ = \textcircled{!}s^\circ$, and $(s+t)^\circ = s^\circ + t^\circ$. Then we extend the translation to translate LP well-formed formulas into LP^* well-formulas such that it fixes sentential letters, commutes with boolean connectives, and $(s{:}A)^\circ = s^\circ{:}A^\circ$.

Lemma 4.5. *If A is an* LP *axiom, then A° is an* LP^* *theorem.*

Proof. Consider the cases of Axiom A1 and A2 as examples. Both $(s{:}(A \to B) \to (t{:}A \to s{\cdot}t{:}B))^\circ$, i.e., $s^\circ{:}(A^\circ \to B^\circ) \to (t^\circ{:}A^\circ \to s^\circ \odot t^\circ{:}B^\circ)$, and $(s{:}A \to !s{:}s{:}A)^\circ$, i.e., $s^\circ{:}A^\circ \to \textcircled{!}s^\circ{:}s^\circ{:}A^\circ$ are LP^* provable by the above lemmas. □

More words on $\textcircled{!}$. First, we can make a caveat to the definition such that $\textcircled{!}p{=}!p$ if p is an LP^* proof term, instead, as in the the original definition, $\textcircled{!}p = e{\cdot}!p$ in which e is the proof term such that $e{:}(p{:}A \to p{:}A)$ is LP^* provable. With the caveat, for any proof term t, super term without $+$, $t^\circ{=}t$. Nonetheless, a merit can be found in our definition.

Our discussion on $\textcircled{!}$ is greatly simplified by our notational convention. For example, in the case of $\textcircled{!}p = \sum e{\cdot}!s_i$ with $p{=}\sum s_i$, the proof term e of $e{\cdot}!s_i$ might be different for each summand s_i of p. But we can make this notational convention not just a convention by extending the proof system LP^* with more axioms such that indeed with the same closed term e, $e{:}(s{:}A \to p{:}A)$ is LP^* provable for all summads s of p, including the case $p{=}s$. Then in this extended system we can define $\textcircled{!}\sum s_i = e \odot \sum !s_i$ and $\textcircled{!}(p{+}q){=}\ \textcircled{!}\ p{+}\ \textcircled{!}\ q$. This latter equation, however, can't be the case if LP^* is not extended in the way just described, since for a summand s of p, the e in the summand $e{\cdot}!s$ of $\textcircled{!}(p+q)$ and the e in the summand $e{\cdot}!s$ of $\textcircled{!}p$ represent different proof terms. This makes constructions involving $\textcircled{!}$ difficult to be recursively analyzed.

Alternatively, we can introduce $\textcircled{!}'$ for the purpose. $\textcircled{!}'p$ is recursively defined such that $\textcircled{!}'t{=}t$, and $\textcircled{!}'(p{+}q){=}e_1{:}\ \textcircled{!}'\ p{+}e_2{:}\ \textcircled{!}'\ q$, where $e_1{:}(p{:}A \to (p{+}q){:}A)$ and $e_2{:}(q{:}A \to (p{+}q){:}A)$ are LP^* theorems for all super terms p, q and well-formed formula A. If we further have $e_1{=}e_2$, then $\textcircled{!}'(p{+}q) = \textcircled{!}'(q{+}p)$. The price to pay for the introduction of $!'$ is that normally more e-like proof terms are needed for $!'p$ than $!p$. For example, for proof terms s, t, u, $!'(s+(t+u))=e_1\cdot!'s+e_2\cdot!'(t+u) =e_1\cdot!'s+e_2\cdot(e_1\cdot!'t+e_2\cdot!'u))$.

Lemma 4.6. *$p{:}A \to !'p{:}p{:}A$ is LP^* provable.*

Proof. By induction on the complexity of super terms. The base case is trivial. For the inductive case, suppose both $\vdash_{\mathsf{LP}^*} p{:}A \to !'p{:}p{:}A$, and $\vdash_{\mathsf{LP}^*} q{:}A \to !'q{:}q{:}A$. Since $e_1{:}(p{:}A \to (p+q){:}A)$ and $e_2{:}(q{:}A \to (p+q){:}A)$ are LP^* theorems, it follows that $!'p{:}p{:}A \to e_1{:}\,!'\,p{:}(p+q){:}A$ and $!'q{:}q{:}A \to e_2{:}\,!'\,q{:}(p+q){:}A$ are LP^* provable. From here, we can derive that $p{:}A \to e_1{:}\,!'\,p{:}(p+q){:}A$ and $q{:}A \to e_2{:}\,!'\,q{:}(p+q){:}A$, and then, by the applications of propositional logic and Axiom A4, $(p+q){:}A \to !'(p+q){:}(p+q){:}A$. □

5 Realization Theorem

Let's call an LP theorem or an LP^* theorem a counterpart of a modal formula, providing if replacing all the terms with modal operator $\Box$, we will get the modal formula. For example $s{:}(P{\to}Q) \to (t{:}P{\to}s{\cdot}t{:}Q)$ for sentential letters P, Q is a counterpart of the formula $\Box(P{\to}Q) \to (\Box P{\to}\Box Q)$. Then we can restate the realization theorem for LP as follows.

Theorem 5.1. *For a modal formula, it is an* $\mathsf{S4}$ *theorem if and only if it has an* LP *counterpart.*

There are two directions in the theorem, with the one from right to left the easy part. So the heavy work is on the justification of the other, and sometimes the title of *realization theorem* is reserved only for this direction of the theorem.

In this section, I will try to use a relatively direct way, employing the resources that I have already had right now, to convince you that LP^* is also a realizable counterpart of $\mathsf{S4}$. By utilizing the lemmas from the last section, my method is some kind of *relative realization*: From the realizability of LP, I will show you the result by analyzing the proofs of the LP counterparts of $\mathsf{S4}$ theorems, instead of providing a constructive argument on the basis of an analysis of Gentzen style $\mathsf{S4}$ proofs, or comparing LP^* models with $\mathsf{S4}$ models to have a semantic argument. However, this method should not be taken for granted, since anyway, if we pay respect to what can be proved, the systems of LP and LP^* are not comparable – we have formulas, such as $(s+t){:}A \to s{:}A \vee t{:}A$, that are provable in LP^* but not in LP, and for super terms p and q, $p \cdot q$ and $!p$ are even not defined in LP^*. What we have done in the last and this section could be viewed as establishing some formal connections between the systems.

Before the main argument, a preliminary work is in order. To facilitate the discussion we will assume that the LP proofs to be analyzed is *injective*, that is, every application of axiom necessitation in the proof is applying a new constant; in other words, in the proof there are no two occurrences of conclusions of axiom necessitation $c_1{:}A$ and $c_2{:}B$ with $c_1{=}c_2$. The feasibility of this assumption is in the freedom of the choice of the constant that the rule of axiom necessitation allows us to do. As for the formal proof, it is in the algorithm of realizing non-circular proofs in [17], which we will skip the details. Roughly speaking, the procedure is that first, by the main result of the article, every $\mathsf{S4}$ theorem has a non-circular S4 proof with the applications of axiom necessitation,[9] and for these non-circular proofs we can always have a way of assigning numeral labels to the modal occurrences in the proofs such that following the order of the numeral labels we are able to step by step substituting a suitable proof term for each modal occurrence and hence turning the whole S4 proofs into LP proofs.[10] The key point to our purpose is that we can assume that every additional modality obtained from an application of axiom necessitation is initial, which means that the suitable proof term for the modal occurrence is not affected by the suitable terms for the other modal occurrences. Then these initial modals can be assigned different number labels and hence can be realized into different constants.[11] Thus without loss of generality we can assume that the LP proof to be dealt with is injective.

Lemma 5.2. *Every* S4 *theorem has an* LP *counterpart whose proof is injective.*

Informally, our proof procedure is to show that for every formula A in an injective LP proof, A°, with $(\cdot)^\circ$ the translation introduced in the last section, is an LP* theorem. One difficulty in the procedure is that we need to deal with formulas of the form $c{:}A(c)$. It's an important and interesting feature in the study of justification logic that such *self-referential* formulas can't be avoided in the process of finding LP counterparts of S4 theorems [5, 18]. The complication in the following lemma is to take care of these formulas.[12]

Lemma 5.3. *If* $A(c_1, ..., c_n)$ *is an* LP *theorem with an injective proof, where all the proof constants in the proof are explicitly indicated, then there are closed terms* $t_1, \ldots, t_n$ *such that* $A^\circ(t_1, \ldots, t_n)$ *is* LP* *provable.*

Proof. I will use the vector notation, i.e. $\vec{t}$ for $t_1, \ldots, t_n$ and $|t| = n$, to simplify the discussion. Suppose $A(\vec{c_1}, \vec{c_2})$ is an LP theorem with an injective proof, where $\vec{c_1}$ are constants introduced by the applications of axiom necessitation in the proof, call them axiom constants of the proof, and $\vec{c_2}$ otherwise, we actually will show by induction on the length of the proof that there are proof terms $\vec{s}$ with $|\vec{s}| = |\vec{c_1}|$ such that for any terms $\vec{t}$ with $|\vec{t}| = |\vec{c_2}|$, $A^\circ(\vec{s}, \vec{t})$ is LP* provable.

For the base case, suppose $A(\vec{c})$ is an LP theorem which has an injective LP proof with length 1, then $A(\vec{c})$ is an LP axiom and certainly no constant c_i in $\vec{c}$ is an axiom constant. By Lemma 4.5, we can see that $A^\circ(\vec{c})$ is an LP* theorem, and then it is not difficult to argue that $A^\circ(\vec{x})$ for variables $\vec{x}$ is an LP* theorem, so $A^\circ(\vec{t})$ for any proof terms $\vec{t}$ are LP* theorem. Now suppose $B(\vec{c_1}, \vec{c_2}, \vec{c_3})$ is derived from $(A \to B)(\vec{c_1}, \vec{c_2}, \vec{c_3})$ and $A(\vec{c_1}, \vec{c_2}, \vec{c_3})$, where $\vec{c_1}$ and $\vec{c_2}$ are the axiom constants for $A \to B$ and A respectively, so $\vec{c_1}$ and $\vec{c_2}$ together are the axiom constants for B. Notice that we can always adjust a proof sequence of B such that the axioms in the subproof of $A \to B$ and in the subproof of A are separated, so the assumption that non-overlaping terms $\vec{c_1}$ and $\vec{c_2}$ as axiom constants for $A \to B$ and A won't lose the generality. By induction hypothesis, there are terms $\vec{s_1}$ and $\vec{s_2}$ such that for any terms $\vec{t_1}$ and $\vec{t_2}$, $(A \to B)^\circ(\vec{s_1}, \vec{t_1}, \vec{t_2})$ and $A^\circ(\vec{t_1}, \vec{s_2}, \vec{t_2})$ are LP* provable. Therefore, for any terms $\vec{t}$, $(A \to B)^\circ(\vec{s_1}, \vec{s_2}, \vec{t})$ and $A^\circ(\vec{s_1}, \vec{s_2}, \vec{t})$ are LP* provable, and so is $B^\circ(\vec{s_1}, \vec{s_2}, \vec{t})$.

[9] Theorem 4.15 and Corollary 4.20 in [17]

[10] Ibid., Theorem 5.8.

[11] Ibid., p. 1330 and Corollary 4.5.

[12] A similar consideration is taken in [7] for dealing with the embeding and equivalence of justification logics.

Finally suppose $B(\vec{c_1}, c, \vec{c_2}) = c{:}A(\vec{c_1}, c, \vec{c_2})$ is derived in an injective LP proof from the application of axiom necessitation on formula $A(\vec{c_1}, c, \vec{c_2})$, where $\vec{c_1}$ are axiom constants of the proof of A and c can't be, since the proof is injective. By induction hypothesis, there are terms $\vec{s}$ for any term u and terms $\vec{t}$, $A^\circ(\vec{s}, u, \vec{t})$ is LP^* provable. Hence, as we have discussed, there is a fixed closed term v such that $v{:}A^\circ(\vec{s}, u, \vec{t})$ is LP^* provable for any u and $\vec{t}$. Then $v{:}A^\circ(\vec{s}, v, \vec{t})$ is LP^* provable. So for any terms $\vec{t}$, $B^\circ(\vec{s}, v, \vec{t})$ is LP^* provable. This completes the proof. □

Theorem 5.4. *For a modal formula, it is an* S4 *theorem if and only if it has an* LP^* *counterpart.*

Proof. The direction from right to left can be proved by induction on the length of a proof of LP^* theorem, and for the other direction, since for a given S4 theorem, there is an LP counterpart $A(\vec{c})$ with an injective proof, by the above lemma, $A^\circ(\vec{t})$ is an LP^* theorem for some terms $\vec{t}$. Then it won't be difficult to check that $A^\circ(\vec{t})$ is an LP^* counterpart of the given S4 theorem. □

6 Arithmetic interpretation of LP^*

The way that we treat $+$ as the union of justifications in the intended epistemic semantics for LP^* can be well carried over to adapt the existing arithmetic semantics for LP to the arithmetic semantics for LP^* in which $+$ functions like the union of proofs. We sketch the idea in this section. First, we consider the arithmetic semantics in [3], and for the technical details, please refer to the paper. Given a constant specification $\mathcal{CS}$, a *$\mathcal{CS}$-interpretation* will interpret each LP formula as a formula in a language of Peano Arithmetic, PA, such that the interpretation of the formulas in $\mathcal{CS}$ will be true in PA. Each *$\mathcal{CS}$-interpretation* will assign to term operators $\cdot$, !, and $+$ their corresponding computable functions satisfying corresponding conditions, and each proof term will correspond to a natural number such that the interpretation of $t{:}A$ is a PA formula $\mathrm{Prf}(k, l)$, where k is the numeral representation of t's corresponding number, l is the numeral representation of the Gödel number of A's corresponding formula in PA, and $\mathrm{Prf}(x, y)$ is a proof predicate satisfies some normality condition which includes the *finiteness* that for every natural number k there are only finitely numbers l such that $\mathrm{Prf}(k, l)$ is true in PA.[13] Then the arithmetic completeness goes in the following way that for a given finite constant specification $\mathcal{CS}$, a formula is $\mathsf{LP}(\mathcal{CS})$ provable if and only if for every *$\mathcal{CS}$-interpretation* it is true in PA under the interpretation. The definition of *$\mathcal{CS}$-interpretation* can be easily adapted to a constant specification $\mathcal{CS}$ of LP^* such that there is no need of assigning a computable function to $+$, but interpreting formulas $(p+q){:}A$ as $p{:}A \vee q{:}A$. We can then have the arithmetic completeness for LP^* in the above sense, with respect to the *$\mathcal{CS}$-interpretation* for LP^*.

The finiteness condition in the above discussion is not material. In [14], a weak arithmetic interpretation is proposed, under which the finiteness is relaxed and hence constant specifications are not limited to being finite. We can then well consider the case of *schematic constant specifications $\mathcal{CS}$* where for each constant c the set $\{A | c{:}A \in \mathcal{CS}\}$ contains all the instances of an axiom scheme

[13] See the formal definition in Definition 6.5 in [3], and the completeness theorem in Corollary 8.9.

or even finite schemes, and hence the total constant specification is one of them. From the proof in [14], we can see that the definition of weak arithmetic interpretation can be well adapted to concerning LP^*, and hence we can also have a weak arithmetic completeness for LP^*.

In the introduction we have complained that LP is a logic of multi-conclusion proofs, but this doesn't imply, as many people might think, that we are advertising a logical system of proofs in which, given that each term justifies only a single formula, $s{:}A \to \neg s{:}B$ with $A \neq B$ or $s{:}A \to \neg s{:}(A \to A)$ is valid, and, nonetheless, can't be the explicit counterpart of a theorem in a normal modal logic.[14] An important point here is that a proof term t can be used to denote more than one proof. A natural case is that t can be used to denote a proof scheme or finite schemes and hence a reading of $t{:}A$ is that A is provable by the proof scheme(s) denoted by t. So in LP^*, we can understand each proof term t as a set of proofs which are bundled together from the beginning and ! will be its proof checker according to the axiom A2. However for a super term $p + q$, ! is not the proof checker, but still as we have discussed above either ⓘ and ⓘ′ can check $p + q$ to see if a formula A is provable by $p + q$.

Either in the arithmetical or weak arithmetic interpretation, proof terms are treated as arbitrary natural numbers, not Gödel numbers of proofs, and the proof predicate used to interpret formulas of the form $t{:}A$ needs not to be any concrete proof predicate $\mathrm{Proof}(x, y)$ defined from a concrete concept of proofs, such as Hilbert style proofs. Concerning what is trying to accomplish by LP, this is an unsatisfactory situation, and the system SLP in [1] is proposed to fix it. Since if, as in SLP, terms are interpreted as Gödel numbers of Hilbert style proofs and + is interpreted as concatenation of multi-conclusion proofs, then certainly we need $(s + t){:}A \leftrightarrow t{:}A \vee s{:}A$ to be valid but it is not LP provable. Hence SLP includes such a principle among others which concerning with the distributivity of $\cdot$ over $+$. But no principles governing the relation between proof checker and + are suggested. So it is not sure if SLP could have the arithmetic completeness with respect to such a interpretation (in [1] only an epistemic-like model of SLP is discussed). However our system seems to have a better chance to be equipped with such kind of semantics, since there is no need to define how the proof $p + q$ is generated from proofs p and q, and we already have the derived relations between + and $\cdot$ and !.

In this paper, we propose a variant justification logical system in which + is treated like the set-theoretical union of proof terms. The axiomatic system with the intended epistemic semantics has been given and the completeness result is established. For the semantics I also suggest a new approach in which an internal proof system is used to record the modeled agent's reasoning ability. This approach accord well with the ideology of justification logic, and we recommend its general adoption for the study of justification logic. We also prove the realization theorem for the system we introduce based on the formal connection between LP and LP^*.

It is indeed that the outlook of our proposed system is not much different from the standard justification logical system. However, the importance of this adjustment should not be overlooked. First, the proposal provides a natural interpretation of the + operator, and this is important in exploring the future

[14] Cf. [12] for a system of such property.

applications of justification logic, and in applying the logic in the right way. Second, as we have discussed, from the beginning the reason for the introduction of the + operator is different from that of the other proof term operators. For the other proof term operators, they all have corresponding modal epistemic axioms. So the function of these operators is indeed the revelation of the hidden justification structure beneath the modal axioms, and hence, in a way, more essential to the study of justification logic than the + operator. Now in our proposal, + is treated differently from the other operators. It is treated as the union of proof terms, not a genuine operation on justificaions. With this treatment, it is then possible in the future when a new justification logical system corresponding to some modal logic is studied, we can limit our attention on the proof term operators other than +.

Finally, as our investigation shows, the tasks that are carried out by LP, including the realization theorem and arithmetic semantics, can be carried out by LP*, which in a way is a logical system of single-conclusion proofs. The point is to give up that + should be a genuine operation on proofs or justification. In the beginning of the project of giving formal arithmetic semantics to the S4 modality, Gödel and Artemov have rightly pointed out that the reason that in the intuitive principle about informal provability:

$$\Box A \to A,$$

we can't read $\Box$ as the formal provability, a Σ_1 formula, in PA, is that under this reading we might not be able to find a witness of the provability formula. And hence making it explicit is the right way to go. But to make it explicit, we don't need to substitute for it a single proof term, but a finite set of proof terms or a super term can do the job well too.

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