Knowledge, Time, and Logical Omniscience

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Abstract. Knowledge's acquisition happens *in* time. However, this feature is not reflected in the standard epistemic logics, e.g. S4 with its possible world semantics suggested by Hintikka in [1], and hence their applications are limited. In this paper we will adapt these normal modal logics to increase their expressive power such that not only what is known is modeled but also when it is to be known is recorded. We supplement each world with an awareness function which is augmented from Fagin-Halpern's to keep track of the time when each formula is to be derived. This provides a new response to the logical omniscience problem. Our work comes from the traditon of study of Justification Logic, also known as Logic of Proofs, LP, introduced by Artemov ([2],[3],[4]). The axiom systems of the models will be given, accompanied with soundness and completeness results.

1 Introduction

To acquire new knowledge, we think, ponder, infer, and so on. All these activities take time. However, this feature of knowledge is not reflected in the standard epistemic logic such as S4, and hence their applications are limited. Since there's no time components helping to indicate when knowledge is acquired by the knower, all knowledge, which is closed under logical consequence, has to be assumed to be known at the one time. This makes the model unrealistic. Our knower is *logical omniscience.* There has been many approaches advocated in the literature to solve this problem. Based on the understanding that the knower modeled by modal epistemic logics knows too much, these approaches provides various mechanisms such as impossible worlds ([5]), awareness functions ([6]), and deduction structures with incomplete deduction rules ([7]), to name some of them, to weaken standard epistemic logics.[1]. Also in [10] the logical omniscience problem is understood as a proof complexity problem. It is suggested that an epistemic logic system is logical omniscience if and only if some knowledge assertion in the system cannot be supported by a feasible size proof. And then it is shown that the *Logic of Proofs*, LP ([3], [4]), now understood as Justification Logic ([2]), is not logical omniscience.

[1] Cf. [8] and chapter 9 in [9] for a survey of the logical omniscience problem.

To solve this problem, however, we choose to adapt the semantics for these standard epistemic logics such that not only will what is known be modeled but also when it is to be known will be recorded. In our point of view, it is not so much a problem for an epistemic logic system which is intended to model an intelligent agents, who are capable of performing logical inferences, to be logical omniscience. But to enhance the applicability, the system has to be accommodated to express the time, or other resources, needed for the knowledge acquisition such that the evolution of knowledge over time can be grasped. For this purpose, we introduce a collection of $\mathtt{S4}^{\Delta}$ logics, in which formulas such as $\mathbf{K}F^i$ are included with i a natural number, meaning formula F is known at time i. The reason why we can have not a single one but a collection of logics corresponding to S4, is that our method is flexible enough to model knowers with different initial knowledges. Some of these logics with richer initial knowledges have been proved having the desired result, the *realization theorem*: every S4 theorem is a theorem in these logics without number labels.

In the literature, there have been many logics proposed combining epistemic and temporal modals, and they have proved very useful.[2] In the semantics of these logics, the time components that the temporal modals range over are better understood as rounds, stages, phases, etc. In each round, the knowers have new information, and their knowledge gets updated. However, the epistemic modals in these logics are essentially modals in normal modal logics, and hence knowers know all the logical consequences of their knowledge within each round. Our approach differs from these in not only using the method of explicit time points, other than temporal modals, to deal with time, but also focusing on increasing the expressive power of standard epistemic logics such that when these logics are applied, we are able to decide, for example, in a round, what can be known by a realistic knower, when whatever time bound is given for that round.

The method we will employ here is basically to extend the use of awareness function, a concept introduced by Fagin and Halpern in [6] to give a possible solution to the logical omniscience problem. In Fagin-Halpern approach, to say that the knower knows some formula at a world means not only is the formula true at all epistemic alternatives to the world, but also the formula should be aware of at the world. In the same paper, they also suggested the possibility to utilize awareness functions so that time can be put into the picture. Our method can be viewed as a direct response to this suggestion. In our usage, not only is what formulas to be aware of in each world provided, but also when will these formulas to be aware of is given. More details will be in the sequel.

The initial of our paper, however, comes from the tradition of study of logics of justification, starting from the *Logic of Proofs*, LP, introduced by Artemov as an explicit proof counterpart of the modal logic S4. The axiom system $\mathtt{S4}^{\Delta}$ of the semantics we are going to present here was first introduced in [17] as an intermediate logic for to discuss the relations between proofs in S4 and proofs in LP. A syntactical proof of the *realization theorem* has been given there. Our model is adapted from Fitting's work [18] on the Kripke-style semantics for

[2] See [11], [12], [13], [14], [15], [16], and also chapters 4 and 8 in [9].

LP, and some terminology is borrowed from there. Except for those necessary alternation from objects representing justifications to time points, we also make modifications both to fit the intuition concerning dealing with time points and for comparisons with other neighboring logics of S4. More words about comparison between S4 and $S4^{\Delta}$ and between $S4^{\Delta}$ and LP will be said later in the paper.

In the final section, we will consider the Δ-style counterparts of other normal modal logics, including logical system with the 5 axiom.

2 Semantics

We first review the possible world semantics in general and the semantics for S4 in particular. The language $L_{\mathbf{K}}$ for S4 and other standard epistemic logics is a extension of the language of propositional logic, i.e., formulas are built up from propositional letters and connectives $\neg$, $\rightarrow$, $\vee$ and $\wedge$, and it has an additional formula formation rule: if $F \in L_{\mathbf{K}}$ then $(\mathbf{K}F)$ is also in $L_{\mathbf{K}}$ (parentheses usually omitted), where $\mathbf{K}F$ means the knower knows F.

A frame is a structure $\langle W, R\rangle$, where W is a non-empty set of possible worlds, and R is a binary relation on W. A standard epistemic model is a structure $\mathcal{M} = \langle W, R, \mathcal{V}\rangle$, where $\langle W, R\rangle$ is a frame, and $\mathcal{V}$ is a valuation from propositional letters to worlds. Then the truth of formulas is defined as (for saving space, we omit the cases for $\wedge$ and $\vee$; nevertheless, they can be redefined by other connectives):

1. $(M, w) \Vdash P$ iff $w \in \mathcal{V}(P)$ for P is a propositional letter;
2. $(M, w) \Vdash \neg F$ iff $w \nVdash F$;
3. $(M, w) \Vdash F \rightarrow G$ iff $w \nVdash F$ or $w \Vdash G$;
4. $(M, w) \Vdash \mathbf{K}F$ iff $w' \Vdash F$ for all $w' \in W$ with wRw'.

In this semantics, the diversity of the knower's epistemic ability is reflected by imposing different conditions on the binary relation. In this paper we mainly consider the semantics which models the knowledge, contrary to the belief, of the knower who has the ability to introspect his/her own knowledge. Then we need the binary relation to be reflexive and transitive. This is the semantics of S4 suggested by Hintikka as an epistemic logic.

This possible world semantic is based on the motto "information is elimination of uncertainty." Even though this motto is quite intuitive, once we directly apply the semantics in a realistic environment, where the knower is supposed to take time to reason, and only a finite amount of time is allowed, our knower knows all logical truth within the amount of time. Most of complaint about this semantics is that the knower modeled knows too much. We, however, consider that there's nothing wrong to suppose that it is possible for the knower to know all these logical truths. But the problem is that different logical truth should take different amount of time to derive, given the knower having some basic logical truths and being able to apply some inference rules. This is just what we are going to do in the following to reveal the hidden temporal relations between knowledge in this semantics.

2.1 Δ-Semantics

The language L_Δ for our $\mathtt{S4}^\Delta$ logics and other Δ-logics is an extension of the language of propositional logic with the formula formation rule: if $F \in L_\Delta$ then $(\mathbf{K}F^i)$ is also in L_Δ, where i is a natural number. The intended meaning of formula $\mathbf{K}F^i$ is that the knower knows F at time i, or the formula F is known at time i (by the knower). We are intendedly not to write the formula as $\mathbf{K}^i F$ because it looks like presupposing that for each i, there is an independent knowledge operator $\mathbf{K}^i$.

The novel tool in our semantics is the awareness function α, which is a partial function mapping formulas in L_Δ to natural numbers. The purpose of the awareness function is to capture the knower's reasoning process by putting formulas in order. One formula is to be aware of later than another because it takes more time to be derived. Some conditions will be imposed on these functions later on to reflect the knower's reasoning ability. A Δ-model is a structure $M = \langle W, R, \{\alpha_w\}, \mathcal{V}\rangle$, where $\langle W, R, \mathcal{V}\rangle$ is a standard epistemic model and $\{\alpha_w\}$ is a collection of awareness functions with index $w \in W$. The truth of formulas in L_Δ is defined as above with the forth condition replaced by:

4'. $(M, w) \Vdash \mathbf{K}F^i$ iff
   - $w' \Vdash F$ for all $w' \in W$ with wRw', and
   - $\alpha_w(F) \leq i$

This condition says that the knower knows F at time i at a world w only if the formula is true at all epistemic alternatives of w and s/he is aware of the formula F before or at i.

2.2 $\mathtt{S4}^\Delta$-Awareness Functions

Awareness functions are grouped by their bases. An awareness base is a tuple $\mathcal{A} = \langle \mathbf{A}, f\rangle$, where the base set $\mathbf{A}$ is a collection of L_Δ formulas, and the base function f is a total function from $\mathbf{A}$ to natural numbers. Formulas in the base set are those to be aware of by the knower either from outside where someone tells him, or from inside where we assume the knower has them inherently, and the base function tells us when the knower is aware of these formulas.

There are three sets of rules that an awareness function should follow. First, given an awareness base $\mathcal{A}$, for an awareness function α based on $\mathcal{A}$ and a formula $A \in \mathcal{A}$, $\alpha(A) \leq f(A)$, i.e., the knower is aware of A before or at $f(A)$ since after $f(A)$ the knower must be aware of the formula. Second, we suppose that our knower has the basic reasoning strength: s/he can do modus ponens and for those formulas in the base set the knower is able to be aware of that s/he knows those formulas. The third set of rules aims to reflect epistemic ability which we assume the knower to possess. Here we need the rule that whatever the formulas that the knower is aware of, s/he is able to be aware of that he knows the formula. Notice that in our usage of being aware of a formula at time i only means that at the moment the knower is aware of the possibility of the formula to be true.

It does not mean that the formula must be true. It is possible for the knower to be aware of mutual contradictory formulas and eventually aware of all formulas. Here's the formal definition.

Definition 1. *Given an awareness base $\mathcal{A} = \langle \mathbf{A}, f \rangle$, an* $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-awareness function *is a partial function from L_Δ to $\mathbb{N}$ satisfying the following conditions ($\alpha(A)\downarrow$ denotes $\alpha(A)$ is defined.):*

1. *Initial condition*
   - *if $A \in \mathbf{A}$, then $\alpha(A)\downarrow$ and $\alpha(A) \leq f(A)$,*
2. *Awareness by deduction*
   a. *if $\alpha(F \to G)\downarrow$ and $\alpha(F)\downarrow$, then*
      - $\alpha(G) \leq \max(\alpha(F \to G), \alpha(F)) + 1$,
   b. *if $A \in \mathbf{A}$ and $f(A) \leq i$, then*
      - $\alpha(\mathbf{K}A^i) \leq i + 1$,
3. ***Inner positive introspection***
   a. *if $F \in L_\Delta$ and $\alpha(F) \leq i$, then*
      - $\alpha(\mathbf{K}F^i) \leq i + 1$.

The condition 2b. is covered by the condition of inner positive introspection. One reason to separate them is that the former condition is more basic than the latter. The other reason is partially syntactical. We will discuss this more in the next section.

In our definition only general epistemic aspects of the knower are reflected in awareness functions. There's no position for the logical strength of the knower, that is, the ability to manipulate logical constants such as *and*, $\wedge$, and *or*, $\vee$. Alternatively, it is determined by the bases. For example, to say a knower knows the logical constant *and*, $\wedge$, is to say that formulas of these kinds, $F \to (G \to (F \wedge G))$ and $(F \wedge G) \to F$, $(F \wedge G) \to F$, are in the base.

Among all $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-awareness functions, there is one playing a special role.

Definition 2. *Given an awareness base $\mathcal{A}$, we say an $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-awareness function $\alpha^*_{\mathcal{A}}$ is* critical *if for any $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-awareness function α, and any $F \in L_\Delta$ with $\alpha^*_{\mathcal{A}}(F)\downarrow$, $\alpha(F) \leq \alpha^*_{\mathcal{A}}(F)$.*

Lemma 1. *For each awareness base $\mathcal{A}$, there exists a unique critical $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-awareness function $\alpha^*_{\mathcal{A}}$.*

2.3 $\mathtt{S4}^{\Delta}$-Semantics and $\mathtt{S4}^{\Delta}$-Awareness Bases

Definition 3. *Given an awareness base $\mathcal{A}$, a Δ-model $M = \langle W, R, \{\alpha_w\}, \mathcal{V} \rangle$ is an* $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-model *if the frame $\langle W, R \rangle$ is reflexive and transitive, and $\{\alpha_w\}$ is a collection of $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-awareness functions and satisfies the* ***monotonicity*** *condition: for any $w\mathcal{R}w'$, $\alpha_{w'}(F) \leq \alpha_w(F)$.*

We say a formula is $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-valid in a $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-model if it is true at all worlds of the model, and a formula is $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-valid, denoted as $\vDash_{\mathtt{S4}^{\Delta}_{\mathcal{A}}} F$ or $\vDash_{\mathcal{A}} F$ for simplicity, if it is valid in all $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-models. The theory of the base, $Th(\mathtt{S4}^{\Delta}_{\mathcal{A}})$, is the set of all $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-valid formulas.

We need more terminology. Given awareness bases $\mathcal{A} = \langle \mathbf{A}, f\rangle$ and $\mathcal{B} = \langle \mathbf{B}, g\rangle$, $\mathcal{B} \subseteq \mathcal{A}$ if $\mathbf{B} \subseteq \mathbf{A}$ and $f(B) = g(B)$ for any $B \in \mathbf{B}$, and $\mathcal{A} \preccurlyeq \mathcal{B}$ if (1) $\mathcal{A} \subseteq \mathcal{B}$ and (2) $\mathbf{B} \subseteq Th(\mathtt{S4}^{\Delta}_{\mathcal{A}})$. For instance, for any $\mathcal{B} = \langle \mathbf{B}, g\rangle$ with $\mathbf{B} \subseteq Th(\mathtt{S4}^{\Delta}_{\emptyset})$, $\emptyset \preccurlyeq \mathcal{B}$, where $\emptyset$ is the empty base. $\preccurlyeq$ is not transitive.

Lemma 2. *For any awareness bases $\mathcal{B} \subseteq \mathcal{A}$, $Th(\mathtt{S4}^{\Delta}_{\mathcal{B}}) \subseteq Th(\mathtt{S4}^{\Delta}_{\mathcal{A}})$.*

Hence for any awareness base $\mathcal{A}$, $Th(\mathtt{S4}^{\Delta}_{\emptyset}) \subseteq Th(\mathtt{S4}^{\Delta}_{\mathcal{A}})$.

Our definition of semantics is very general. For any awareness base $\mathcal{A}$, even if it might contain formulas contradictory to each other, $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-models are defined. However, we are interested in the bases which contain only valid formulas, and can be used to characterize the logical strength of knowers by the number of logical truths they have. We need a constructive way to provide a general definition of bases of this kind.

Definition 4. *We say an awareness base $\mathcal{A} = \langle \mathbf{A}, f\rangle$ is an $\mathtt{S4}^{\Delta}$-awareness base if it satisfies one of the following three conditions:*

1. *$\mathcal{A} = \emptyset$.*
2. *There are awareness bases $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ such that $\emptyset = \mathcal{A}_0 \preccurlyeq \mathcal{A}_1 \preccurlyeq \cdots \preccurlyeq \mathcal{A}_n = \mathcal{A}$.*
3. *There is a collection of awareness bases $\{\mathcal{A}_i\}_{i\in\mathbb{N}}$ with $\mathcal{A}_i = \langle \mathbf{A}_i f_i\rangle$ such that $\emptyset = \mathcal{A}_0 \preccurlyeq \mathcal{A}_1 \preccurlyeq \mathcal{A}_2 \preccurlyeq \cdots$, and $\mathcal{A} = \bigcup \mathcal{A}_i$, that is, $\mathbf{A} = \bigcup \mathbf{A}_i$ and for any $A \in \mathbf{A}_i$, $f(A) = f_i(A)$.*

Then it is not difficult to check that if $\mathcal{A}$ is an $\mathtt{S4}^{\Delta}$-awareness base, for every $A \in \mathcal{A}$, $\vDash_{\mathcal{A}} A$.

We say an awareness base is finite if its base set is finite.

Lemma 3. *Given an $\mathtt{S4}^{\Delta}$-awareness base $\mathcal{A}$ and a formula $F \in L_{\Delta}$, $\vDash_{\mathcal{A}} F$ if and only if there is a finite $\mathtt{S4}^{\Delta}$-awareness base $\mathcal{B} \subseteq \mathcal{A}$, $\vDash_{\mathcal{B}} F$.*

This lemma can be shown by first proving the compactness theorem of our semantics, or is an immediate corollary of the completeness result that we will give later when axiom systems of our semantics are introduced. There is an interesting result about the critical awareness function.

Lemma 4. *For each $\mathtt{S4}^{\Delta}$-awareness base $\mathcal{A}$, if $\alpha^*_{\mathcal{A}}(F)\downarrow$ then $\vDash_{\mathcal{A}} F$.*

The statement is not true for an arbitrary $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-awareness function. Before closing this subsection, we consider several conditions on collections of awareness functions which related to positive introspection, in contrast with conditions we will introduce later related to the negative introspection.

Definition 5. *We say a collection of awareness functions satisfies the* ***inner positive introspection*** *condition if every element of the collection satisfies the condition.*

Definition 6. *Given a model $M = \langle W, R, \{\alpha_w\}, \mathcal{V}\rangle$, we say the collection $\{\alpha_w\}$ satisfies the **outer positive introspection** condition if for any w with $(M, w) \Vdash \mathbf{K}F^i$, $\alpha_w(\mathbf{K}F^i) \leq i + 1$.*

This condition says that if at some world the knower knows some formula at some time, then at the world he is aware of the formula one time unit latter. Though, this condition is derivable.

Fact. If a collection of awareness function satisfies the inner positive introspection condition, then the collection satisfies the outer positive introspection condition for any model the collection belongs to.

Now we put all the conditions relative to the positive introspection together.

Definition 7. *Given a model $M = \langle W, R, \{\alpha_w\}, \mathcal{V}\rangle$, we say the collection $\{\alpha_w\}$ is **positive regular** if it is a monotonic and satisfies both inner and outer positive introspection conditions.*

2.4 More on $\mathtt{S4}^{\Delta}$-Awareness Bases

In our semantics, knowers with different awareness bases are regarded as having different epistemic strength, and modeled differently. The smallest awareness base is the empty base. We also can have a maximal awareness base. An $\mathtt{S4}^{\Delta}$-awareness base $\mathcal{A} = \langle \mathbf{A}, f\rangle$ is maximal if $Th(\mathtt{S4}^{\Delta}_{\mathcal{A}}) \subseteq \mathbf{A}$. Let $\{\mathcal{A}_i\}_{i\in\mathbb{N}}$ be a collection of awareness bases with $\mathcal{A}_0$ the empty base and $\mathcal{A}_{i+1} = \langle \mathbf{A}_{i+1}, f_{i+1}\rangle$ such that $\mathbf{A}_{i+1} = Th(\mathtt{S4}^{\Delta}_{\mathcal{A}_i})$, and $f_{i+1}(A) = f_i(A)$ for any $A \in \mathbf{A}_i$, and let $\mathcal{A} = \bigcup \mathcal{A}_i$. Then by using Lemma 3, it can be proved that $\mathcal{A}$ is a maximal $\mathtt{S4}^{\Delta}$-awareness base.

Notice that there is more than one maximal awareness base. Different base functions will give us different maximal awareness bases. Call an awareness base with the constant function 0 principal, and let $\mathcal{A}$ be the principal maximal awareness base. Then if F is a $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-valid formula, so is $\mathbf{K}F^0$.

Obviously, knowers with maximal awareness bases are not realistic. Several other types of awareness bases have more intuitive appealing. Here's the list. Let $\mathcal{A}$ be an $\mathtt{S4}^{\Delta}$-awareness base.

1. There is a maximal base $\mathcal{B}$ such that $Th(\mathtt{S4}^{\Delta}_{\mathcal{B}}) \subseteq Th(\mathtt{S4}^{\Delta}_{\mathcal{A}})$.
2. For any $F \in Th(\mathtt{S4}^{\Delta}_{\mathcal{A}})$, $\alpha^*_{\mathcal{A}}(F) \downarrow$.
3. For any $F \in Th(\mathtt{S4}^{\Delta}_{\mathcal{A}})$, $\{\alpha(F)\}$ is bounded.
4. For any $F \in Th(\mathtt{S4}^{\Delta}_{\mathcal{A}})$, $\mathbf{K}F^i \in Th(\mathtt{S4}^{\Delta}_{\mathcal{A}})$ for some i.

The four definitions of these properties are from four different concerns. The first one concerns the relationship between the theory of the base and the theory of a maximal one. The statement in the second item is the converse of of Lemma 4, concerning critical awareness function. The third one concerns the awareness functions in general and the last one is about the theory of the base itself. Now what really interests us is that these awareness bases are all equivalent.

Theorem 1. *All the four properties of an* $\mathtt{S4}^{\Delta}$*-awareness base* $\mathcal{A}$ *defined above are equivalent.*

Definition 8. *An awareness base with one of the above property is called* full.

These full awareness bases turn out to play a special role for the realization theorem. In the next section we will see a concrete example of a full base.

3 Axiom Systems

Given an $\mathtt{S4}^{\Delta}$-awareness base $\mathcal{A} = \langle \mathbf{A}, f \rangle$, the following is the axiom system $\mathtt{S4}^{\Delta}_{\mathcal{A}}$.

Definition 9.
Axioms
A0 classical propositional axiom schemes
A1 $\mathbf{K}(F \to G)^i \to \mathbf{K}F^j \to \mathbf{K}G^k \quad i, j < k$
A2 $\mathbf{K}A^i \to \mathbf{K}(\mathbf{K}A^i)^j \quad i < j \;$ *if* $A \in \mathbf{A}$ *and* $f(A) \le i$
A3 $\mathbf{K}F^i \to \mathbf{K}F^j \quad i < j$
A4 $\mathbf{K}F^i \to \mathbf{K}(\mathbf{K}F^i)^j \quad i < j$
A5 $\mathbf{K}F^i \to F$
Inference Rule
R1 $\vdash G$*, if* $\vdash F \to G$ *and* $\vdash F$ *"modus ponens"*
R2 $\vdash \mathbf{K}A^i$ *if* $A \in \mathbf{A}$ *and* $f(A) \le i$ *"*$\mathcal{A}$*-necessitation"*

We use $\vdash_{\mathtt{S4}^{\Delta}_{\mathcal{A}}} F$, or $\vdash_{\mathcal{A}} F$ for simplicity, to denote that F is a theorem in $\mathtt{S4}^{\Delta}_{\mathcal{A}}$.

Theorem 2. *For any awareness base* $\mathcal{A}$*,* $\vDash_{\mathcal{A}} F$ *if and only if* $\vdash_{\mathcal{A}} F$*.*

The proof is given in the appendix.

When our $\mathcal{A}$ is empty, the A2 axiom and the R2 rule are void. When $\mathcal{A}$ is maximal, the clause "$A \in \mathcal{A}$" can be replaced by "$\vdash A$", and when the base is principal, "and $f(A) \le i$" can be removed. The most interesting awareness base will be the one containing all axioms. This one will be full.

Lemma 5. *Given an awareness base* $\mathcal{A} = \langle \mathbf{A}, f \rangle$*, if* $\mathbf{A}$ *contains all axiom instance of the system,* $\mathcal{A}$ *is full.*

Proof. With the completeness and soundness results above, it is sufficient to prove that if $\vdash_{\mathcal{A}} F$, then $\vdash_{\mathcal{A}} \mathbf{K}F^i$ for some i (hence if $\vDash_{\mathcal{A}} F$, $\vDash_{\mathcal{A}} \mathbf{K}F^i$). We prove the statement by induction on the length of the proof of F. Suppose F is an axiom, then by $\mathcal{A}$-necessitation, $\vdash_{\mathcal{A}} \mathbf{K}F^i$ for $i \ge f(A)$. If G is derived from $F \to G$ and F, by induction hypothesis $\vdash_{\mathcal{A}} \mathbf{K}(F \to G)^i$ and $\vdash_{\mathcal{A}} \mathbf{K}F^j$. Then using axiom A1, we have $\vdash_{\mathcal{A}} \mathbf{K}G^k$ for $k > i, j$. If $\mathbf{K}F^i$ is derived by $\mathcal{A}$-necessitation, by applying the axiom A2, $\vdash_{\mathcal{A}} \mathbf{K}(\mathbf{K}F^i)^j$ for $j > i$[3].

[3] This result is similar to Artemov's Internalization Theorem in LP [4].

Definition 10. *We say an awareness base $\mathcal{A}$ is* axiomatically appropriate *if its base set includes all axiom instances of the schemes in* $\mathtt{S4}^{\Delta}_{\mathcal{A}}$.

When $\mathcal{A}$ is axiomatically appropriate, the clause "$A \in \mathcal{A}$" can be replaced by "A is an axiom" in the axiom system.

Notice that the A1 and A2 axioms are needed in the proof of Lemma 5. In our $\mathtt{S4}^{\Delta}$ axiom systems, all axioms except for the A2 axioms are $\mathtt{S4}^{\Delta}_{\emptyset}$-valid, and since all the functions of the A2 axioms can be replaced by the A4 axioms, thus an axiomatically appropriate base without the A2 axioms is still full. This makes it possible to have a full $\mathtt{S4}^{\Delta}$-awareness base $\mathcal{A}$ with $\emptyset \preccurlyeq \mathcal{A}$. But on the other hand, if we consider the Δ-counterpart of other normal modal logics, especially those without the A4 axioms, then the A2 axioms, which is corresponding to the 2b. condition of Definition 1, cannot be excluded from axiom appropriate bases to have full awareness bases.

4 Realization Theorem and Relations to LP

As mentioned earlier, our attitude to the problem of logical omniscience is that the standard epistemic logics, such as S4, do not model a knower who knows too much, but, instead, lack the expressive power such that when it is applied, we have to ascribe all logical truths as knowledge to the knower. To justify our claim, relations between S4 and $\mathtt{S4}^{\Delta}$ logics should be built. There are two directions of the relations. One is trivial. Roughly speaking, if we drop all the number labels from an $\mathtt{S4}^{\Delta}_{\mathcal{A}}$ proof, we will have an S4 proof. This implies that every $\mathtt{S4}^{\Delta}_{\mathcal{A}}$ theorem is an S4 theorem with time labels. For the other direction, in [17] it has been constructively proved that, for the principal axiomatically appropriate awareness base $\mathcal{A}$, every theorem in S4 can be translated to a theorem in $\mathtt{S4}^{\Delta}_{\mathcal{A}}$ by adding suitable time labels to modal formulas. This theorem is called the *realization theorem*. A sketch of the proof of this theorem is given at the end of this paper. The semantic proof of this theorem is in progress. From the experience of working on LP, the fullness of an awareness base would be sufficient for the realization theorem to hold. Now we can conclude that theorems in S4 provides us the relations between the knower's knowledges without considering the relations between the time points indicating when the knowledges is known, which are hidden in the S4 theorems and revealed in the theorems' Δ-counterparts in $\mathtt{S4}^{\Delta}_{\mathcal{A}}$.

LP now is one of the family of *Justification Logics*. It is not only a logic passing the logical omniscience test, as we mentioned before, but also a logic with the justificatory complexity of knowledge acquisition is recorded. It has formula atoms like $t\!:\!F$ to mean t is a justification of F with t being an justification object. A efficient translation between proofs in $\mathtt{S4}^{\Delta}$ and proofs in LP has been given in [17]. The translation reflects an informal relations between justifications and time. Justifications need time, and we gain knowledge through time by giving justifications. LP has a more refined framework than $\mathtt{S4}^{\Delta}$. With justifications explicitly expressed, we can not only identify the temporal order of what we know but also trace the reasoning history of our knowledge. However, reasoning in $\mathtt{S4}^{\Delta}$ is more intuitive and working on natural numbers with their linearity is easier than directly dealing with justification objects.

5 Variations

Our logic $\mathtt{S4}^{\Delta}$ is adapted from S4, both semantically and syntactically. We add an $\mathtt{S4}^{\Delta}$-awareness function to each world in a S4 model to get the $\mathtt{S4}^{\Delta}$-semantics and add number labels to axiom schemes to have Δ-style axiom systems. By similar manner we can have Δ-counterparts of the neighboring logics of S4, semantically and syntactically. Let $M = \langle W, R, \{\alpha_w\}, \mathcal{V}\rangle$ be a Δ-model. In this section, awareness functions are not supposed to satisfy the inner positive introspection condition. The following is a table of the conditions on frames and collections of awareness functions of models that the complete and sound semantics of these sublogics of $\mathtt{S4}^{\Delta}$ should satisfy.

	$\langle W, R\rangle$	$\{\alpha_w\}$
$\mathtt{K}^{\Delta}$	no condition	no condition
$\mathtt{KT}^{\Delta}$	reflexive	no condition
$\mathtt{K4}^{\Delta}$	transitive	positive regular
$\mathtt{KT4}^{\Delta}$	transitive and reflexive	positive regular

For these Δ-logics, when an awareness base is axiomatically appropriate with respect to their axiom systems, the base is full. Here the axiom A2 plays his role.

Now add the 5 axiom[4]: "$\neg\mathbf{K}F^i \rightarrow \mathbf{K}\neg\mathbf{K}F^j$ for $i < j$." into the picture. We say an awareness function α satisfies the **inner negative introspection** condition if $\alpha(\neg\mathbf{K}F^i) \leq i+1$, provided $\alpha(F) \not\leq i$, and a collection of awareness functions satisfies the condition if every element of the collection satisfies the condition. Given a model $M = \langle W, R, \{\alpha_w\}, \mathcal{V}\rangle$, we say the collection $\{\alpha_w\}$ is **anti-monotonic** if for any $w\mathcal{R}w'$, whenever $\alpha_w(F) \not\leq i$, then $\alpha_{w'}(F) \not\leq i$, and we say the collection $\{\alpha_w\}$ satisfies the **outer negative introspection** condition if $\alpha_w(\neg\mathbf{K}F^i) \leq i+1$, provided $(M, w) \Vdash \neg\mathbf{K}F^i$. Finally, we call the collection is **negative regular** if the collection is anti-monotonic and satisfies both the inner and outer negative introspection conditions. Then for those systems with the 5 axiom, in their complete and sound models, the frame should be *Euclidean*, and the collection of awareness functions is negative regular.

Fact. If a collection of awareness function in a model satisfies the outer negative introspection condition, then the collection satisfies the inner negative introspection condition.

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[4] This work here is adapted from [20] and [21], more related to [20], but different.

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Appendix

Some Proofs

Proof of Lemma 1

Proof. Let $a = \min\{j \mid f(A) = j$, for every $A \in \mathbf{A}\}$ with f the base function. The construction is straightforward. First, let $\alpha^*_{\mathcal{A}}(F) = a$ for any $f(F) = a$. Then suppose the construction of $\alpha^*_{\mathcal{A}}$ have been completed up to n, we just define $\alpha^*(G) = n + 1$ for whatever formula G needed to be defined. That is if $\alpha^*_{\mathcal{A}}(G)$ is not defined yet when, say, $\alpha^*_{\mathcal{A}}(F \to G)$ and $\alpha^*_{\mathcal{A}}(F)$ is defined, or $G \in \mathbf{A}$ with $f(G) = n + 1$, define $\alpha^*_{\mathcal{A}}(G) = n + 1$. Continue this process we have the critical awareness function.

Proof of Theorem 1

Proof. We prove the directions from 1 to 4 and 2 to 1, The directions form 4 to 3 and 3 to 2 are trivial. Suppose $\vDash_{\mathcal{A}} F$ and there is a maximal base $\mathcal{A}'$ such that $\vDash_{\mathcal{A}'} F$, then $F \in \mathcal{A}'$. Since $\vDash_{\mathcal{A}'} \mathbf{K}F^i$ for some i, $\vDash_{\mathcal{A}} \mathbf{K}F^i$. This proves the direction from 1 to 4.

Now we prove the direction from 2 to 1. Suppose for every $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-valid formula F, $\alpha^*_{\mathcal{A}}(F)$ is defined. We define $\mathcal{A}' = \langle \mathbf{A}', f' \rangle$ with $\mathbf{A}' = \{F \mid \vDash_{\mathcal{A}} F\}$ and $f'(F) = \alpha^*_{\mathcal{A}}(F)$. Since for any $F \in \mathbf{A}'$ and any $\mathcal{A}$-awareness function α, $\alpha(F) \leq f'(F)(= \alpha^*_{\mathcal{A}}(F))$, α is an $\mathcal{A}'$-awareness function. Hence every $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-model is a $\mathtt{S4}^{\Delta}_{\mathcal{A}'}$-model and every $\mathtt{S4}^{\Delta}_{\mathcal{A}'}$-valid formula is an $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-valid formula. By the definition of $\mathcal{A}'$, every $\mathtt{S4}^{\Delta}_{\mathcal{A}}$-valid formula is in $\mathcal{A}'$. $\mathcal{A}'$ is maximal.

Proof of Theorem 2

Proof. The soundness part is straightforward. For the completeness part, we construct a model composed of maximal $\mathcal{A}$-consistent sets. A set S of formulas in L_{Δ} is said to be $\mathcal{A}$-consistent if there is no finite subset $\{F_1, \ldots, F_n\}$ of S such that $\vdash_{\mathcal{A}} \neg(F_1 \wedge \ldots \wedge F_n)$. The construction of a maximal such set is by the standard Lindenbaum construction. Let W be the set of all maximal $\mathcal{A}$-consistent sets, and for any Γ, $\Gamma' \in W$, $\Gamma R \Gamma'$ if and only if $\Gamma^\sharp \subseteq \Gamma'$ where $\Gamma^\sharp = \{F \mid \mathbf{K}F^i \in \Gamma\}$. $\alpha_\Gamma(F) = \min\{i \mid \mathbf{K}F^i \in \Gamma\}$ and $\mathcal{V}(P) = \{\Gamma \mid P \in \Gamma\}$. We claim this $M = \langle W, R, \{\alpha_w\}, \mathcal{V} \rangle$ is a $\mathtt{S4}^{\Delta}_{\mathcal{A}}$ model. R is transitive and reflexive because of the axioms A2 and A4. α_Γ satisfy the initial condition due to the $\mathcal{A}$-necessitation rule, and it is not difficult to check that α_Γ satisfies all other conditions by applying these conditions' corresponding axioms. The collection of these awareness functions also satisfies the monotonicity condition. When $\alpha_\Gamma(F) = i$, $\mathbf{K}F^i \in \Gamma$. Since $\vDash_{\mathcal{A}} \mathbf{K}F^i \to \mathbf{K}(\mathbf{K}F^i)^j$, $\mathbf{K}(\mathbf{K}F^i)^j \in \Gamma$, so $\mathbf{K}F^i \in \Gamma'$ for any $\Gamma^\sharp \subseteq \Gamma'$. $\alpha_{\Gamma'}(F) \leq i$.

Then is the truth lemma, that is, for every Γ, $F \in \Gamma$ if and only if $(M, \Gamma) \Vdash F$. The proof is by inductin on the complexity of formulas. Most of cases are trivial. We prove the modal case. If $(M, \Gamma) \Vdash \mathbf{K}F^i$ then $\alpha_\Gamma(F) \leq i$, $\mathbf{K}F^i \in \Gamma$. For the other direction, if $\mathbf{K}F^i \in \Gamma$, $\alpha_\Gamma(F) \leq i$ and for any $\Gamma^\sharp \subseteq \Gamma'$, $F \in \Gamma'$, so by induction $(M, \Gamma') \Vdash F$. Hence $(M, \Gamma) \Vdash \mathbf{K}F^i$. Now suppose F is not provable in $\mathtt{S4}^{\Delta}_{\mathcal{A}}$, $\neg F$ is $\mathcal{A}$-consistent. $(M, \Gamma) \nVdash F$ with Γ a maximal $\mathcal{A}$-consistent set containing $\neg F$.

The Realization Theorem

The realization procedure in [17] is the following. A Δ-style cut-free Gentzen system $\mathtt{S4}^{\Delta}G^{-}$ corresponding to a cut-free Gentzen system $\mathtt{S4}G^{-}$ of $\mathtt{S4}$ is introduced. Every rule in the system is derivable in $\mathtt{S4}^{\Delta}_{\mathcal{A}}$ with $\mathcal{A}$ the principle maximal awareness base. Then it is shown that every proof in $\mathtt{S4}G^{-}$ can be turned into a proof in $\mathtt{S4}^{\Delta}G^{-}$ by supplementing suitable natural number labels to every modal formulas. So it follows that every $\mathtt{S4}$ theorem can be converted to an $\mathtt{S4}^{\Delta}_{\mathcal{A}}$ theorem.

A sequent $\Gamma \Rightarrow \Gamma'$ is a pair of finite multisets Γ, Γ' of formulas. It is convenient to view a sequenct as a formula $C_1 \to (\ldots \to (C_n \to \bigvee \Gamma')\ldots)$ here. Given a multiset $\Gamma = \{C_i\}$ of formulas in $L_{\mathbf{K}}$, $\mathbf{K}\Gamma = \{\mathbf{K}C_i\}$. Given a multiset $\Gamma = \{C_i\}$ of formulas in L_{Δ}, $\mathbf{K}\Gamma^{\iota} = \{\mathbf{K}C_i^{j_i}\}$, for j_i a number in the multiset ι. $|\Gamma|$ is the number of formulas in Γ. Here's the Gentzen systems.

Definition 11 ($\mathtt{S4}G^{-}$).
The only axiom is that $P \Rightarrow P$, for a propositional letter P.
The rules for weakening (W) and constraction (C)

$$LW\,\frac{\Gamma \Rightarrow \Gamma'}{A, \Gamma \Rightarrow \Gamma'} \qquad RW\,\frac{\Gamma \Rightarrow \Gamma'}{\Gamma \Rightarrow \Gamma', A}$$

$$LC\,\frac{A, A, \Gamma \Rightarrow \Gamma'}{A, \Gamma \Rightarrow \Gamma'} \qquad RC\,\frac{\Gamma \Rightarrow \Gamma', A, A}{\Gamma \Rightarrow \Gamma', A}$$

The classical logical rules

$$L\neg\,\frac{\Gamma \Rightarrow \Gamma', A}{\neg A, \Gamma \Rightarrow \Gamma'} \qquad R\neg\,\frac{\Gamma, A \Rightarrow \Gamma'}{\Gamma \Rightarrow \Gamma', \neg A}$$

$$L\!\to\,\frac{\Gamma \Rightarrow \Gamma', A \quad B, \Gamma \Rightarrow \Gamma'}{A \to B, \Gamma \Rightarrow \Gamma'} \qquad R\!\to\,\frac{A, \Gamma \Rightarrow \Gamma', B}{\Gamma \Rightarrow \Gamma', A \to B}$$

The modal rules are

$$R\mathbf{K}\,\frac{A, \Gamma \Rightarrow \Gamma'}{\mathbf{K}A, \Gamma \Rightarrow \Gamma'} \qquad R\mathbf{K}\,\frac{\mathbf{K}\Gamma \Rightarrow A}{\mathbf{K}\Gamma \Rightarrow \mathbf{K}A}$$

Definition 12 ($\mathtt{S4}^{\Delta}G^{-}$).
The language for $\mathtt{S4}^{\Delta}G^{-}$ is L_{Δ}. It is the $\mathtt{S4}G^{-}$ with the following Δ-modal rules

$$L\mathbf{K}\,\frac{A, \Gamma \Rightarrow \Gamma'}{\mathbf{K}A^{i}, \Gamma \Rightarrow \Gamma'}\,, \quad \textit{for any } i$$

$$R\mathbf{K}\,\frac{\mathbf{K}\Gamma^{\iota} \Rightarrow A}{\mathbf{K}\Gamma^{\iota} \Rightarrow \mathbf{K}A^{i}}\,, \quad \textit{for any } i > \max(\iota) + |\Gamma| + 1, \textit{ when } |\Gamma| \neq 0, \textit{ and}$$
for any i when $|\Gamma| = 0$

Lemma 6. *Every rule in the system is derivable in $\mathtt{S4}^{\Delta}_{\mathcal{A}}$ with $\mathcal{A}$ the principle maximal awareness base.*

Proof. It is sufficient to check the Δ-modal cases. The case for the left modal rule is also trivial. By induction, it can be checked that for $|\Gamma| > 0$ and $i > \max(\iota, e) + |\Gamma| + 1$, $\mathbf{K}(\mathbf{K}\Gamma^{\iota} \Rightarrow A)^{e} \to (\mathbf{K}\Gamma^{\iota} \Rightarrow \mathbf{K}A^{i})$ is provable in $\mathtt{S4}^{\Delta}_{\mathcal{A}}$ with $\mathcal{A}$ the principle maximal awareness base. Then it follows that the right modal rule is derivable.

Lemma 7. *Every* $\mathtt{S4}G^-$ *proof is a proof of* $\mathtt{S4}^{\Delta}G^-$ *without number labels.*

Proof. Call the formulas with the modal operator **K** as the main connectives as m-formulas. Let all negative m-formula occurrences have label 0, and all positive m-formula occurrences have the label equal to the nubmer of formula occurrences in the $\mathtt{S4}G^-$ proof. Then the condition for the right Δ-modal rule will be satisfied. So the resulting sequent tree is a proof in $\mathtt{S4}^{\Delta}G^-$.

In [17], a procedure coverting a proof in $\mathtt{S4}^{\Delta}_{\mathcal{A}}$ with $\mathcal{A}$ the principal maximal awareness base to a proof in $\mathtt{S4}^{\Delta}_{\mathcal{A}'}$ with $\mathcal{A}'$ the principal axiomatically appropriate awareness base is also given . The statement is jsutified.

Theorem 3. *Given* $\mathcal{A}$ *the principal axiomatically appropriate awareness base, every* $\mathtt{S4}$ *theorem is a* $\mathtt{S4}^{\Delta}_{\mathcal{A}}$ *theorem without number labels.*