Fundamenta Informaticae XX (2010) 1–18
IOS Press

Knowledge, Time, and the Problem of Logical Omniscience

Ren-June Wang
Computer Science
CUNY Graduate Center
365 Fifth Avenue, New York, NY 10016
rwang@gc.cuny.edu

Abstract. It is well known that Modal Epistemic logic (MEL) suffers from the problem of logical omniscience. In this paper, we will argue that in order to solve the problem, the temporal dimension of knowledge has to be revealed and following this analysis, we present a general epistemic framework, timed Modal Epistemic Logic (tMEL), modified from MEL, such that the time at which a formula is known by an agent based on his reasoning procedure is explicitly stated. With the help of the additional temporal devices, we are able to determine what is actually known by the agent within a reasonable time of reasoning. The discussions will focus on tS4, the tMEL counterpart of S4, but the method can be uniformly generalized to the study of other tMEL logics. Both the semantics and axiomatic proof systems will be provided, accompanied by soundness and completeness results, and other technical features of tMEL are also examined. This work originates from the study of *Justification Logic*, which shapes many aspects of this paper, and is also a direct response to the request to utilize the use of *awareness functions* such that time can be added to the picture. A generalized awareness function is employed in the semantics to trace when a formula is deduced.

1. Introduction

Modern epistemic logic started in the middle of the last century when von Wright, a Swedish-Finnish philosopher, introduced the first deductive epistemic system, and later another Finn, Hintikka, proposed a semantical analysis in *Knowledge and Belief*, which is arguably the most influential work in epistemic logic.[1] Epistemic notions in these works are modeled on modal logic, and the semantic framework suggested by Hintikka is the familiar *possible world semantics*. The basic idea of the semantics is "the old adage *information means elimination of uncertainty*,"[2] i.e. that someone knows a formula in a world if and only if the formula is true in all epistemic alternatives indistinguishable, from the knower's point of

[1]See von Wright [36], and Hintikka [18].
[2]See quote in [19, p. 64].

view, to the world in question. This modal approach to epistemic logic (MEL), with its possible world semantics, provides an intuitive and mathematically elegant way of representing and reasoning about knowledge, but from the beginning, philosophers has noticed that MEL presupposes agents with unrealistic reasoning ability,[3] a problem normally known as the *problem of logical omniscience*. Since the eighties of the last century, MEL has found applications in various fields such as Artificial Intelligence, Computer Science, and Economics. In some of these applications (e.g. distributed systems), the logical omniscience problem is not a concern, but for the most part, the problem is an obstacle to further applications. Alternative approaches have been suggested in the literature, and in this paper, we add to the list another one, which, nevertheless, has some favorable features not shared by the other approaches.

Consider the following *Rule of Knowledge Closure*, which is derivable both semantically and syntactically, in all the standard MEL logics, basically the *normal modal logics*:

$$(1) \quad \frac{\vdash \phi \to \psi}{\vdash \mathbf{K}\phi \to \mathbf{K}\psi} \quad \text{(Knowledge Closure)}$$

It states that for any logically true implication, if the agent knows the premise, the conclusion is also known. Obviously, none of normal agents have reasoning ability of this kind. Human beings and intelligent machines are unlikely to know all the logical consequences of their knowledge. For some logical implications, the conclusions are just too removed from their premises and agents of normal reasoning ability are not able to complete the reasoning processes within a reasonable time. One way to remove this unrealisticity is to employ some weaker logics at hand as epistemic logics. For example, it is suggested to take Montague-Scott's *neighborhood semantics* as the alternative epistemic model,[4] in which rule of inference (1) is not justified. The problem with this approach is that it does not resolve logical omniscience thoroughly but only alleviates the problem to some extent. The agents modeled by neighborhood semantics are still unrealistic. They are able to know all of the logical equivalences of whatever they know.[5] An epistemic model with two notions of belief, *explicit* and *implicit*, is introduced in [23], which is an influential work in the current study of the logical omniscience problem. The idea is that explicit belief is the belief we actually have, while implicit belief is the logical consequence of explicit belief. An agent in this model does not (explicitly) believe all the logical consequences of his belief in the ordinary sense, that is, in the sense of classical logic. However, it has been remarked that the agent does believe all the logical consequences of his belief in terms of a nonstandard logic, the relevance logic,[6] and this is still not something that a normal agent is able to achieve.

The dissatisfaction of these approaches leads to the consideration of epistemic models in which Rule of Knowledge Closure is completely invalid. An agent knowing none of the logical consequences of his knowledge is modeled and hence no single logical implication or set of logical implications can be applied by this rule. At present, this condition seems to become the standard criterion for an epistemic model that qualifies as a solution to the logical omniscience problem. Many well-known approaches share this feature. These include the syntactical approach, which abandons possible world semantics but identifies knowledge as simply a set of sentences, or a set of sentences derived from a set of deductive

[3]See [18, p. 36]

[4]See the semantics in [25], [33], and also in [8]. The suggestion is made in [12].

[5]In neighborhood semantics, $\frac{\vdash \phi \leftrightarrow \psi}{\vdash \mathbf{K}\phi \leftrightarrow \mathbf{K}\psi}$ is valid.

[6]See [23] and [34] for this kind of remark.

rules ([10], [26], [21]); the impossible world approach, which is modified from possible world semantics such that each world could have impossible (inconsistent) worlds as its epistemic alternatives ([31]); and the awareness approach, in which, following the idea in [23], implicit knowledge is defined in terms of possible world semantics and a syntactical structure, called the *awareness function*, is introduced to distinguish the explicit knowledge from the implicit ([11]).[7]

Even though this type of solution to the logical omniscience problem is current prevalent, it is highly problematic. There is no doubt that epistemic logic should be established without the assumption of logical omniscience, Normal agents, such as human beings or machines with computability, don't have the reasoning ability to know all the logical consequences of their knowledge. But this doesn't mean that they don't have any logical ability at all. The epistemic logics suggested in this type of solution are too weak; agents are not presupposed to be able to perform any derivations, even as simple as derivations from conjunctions to their conjuncts. From the beginning, we observe that there is logical structure in our usage of epistemic terms. This demonstrates that we possess the ability to reason logically and hence epistemic logic was introduced. We then found that the most intuitive and elegant epistemic model is unrealistic, and started to search for alternative approaches to solve the problem. But the solutions we have so far either still commit us to some kind of logical omniscience, or to model agents without logical ability at all. We need a new type of approach to deal with the problem.

Now let's go back to the source of the problem. As mentioned earlier, the reason that a normal agent does not know all the logical consequences of his knowledge is that some consequences are just computationally too far to reach. It is not because normal agents have weaker reasoning mechanisms or lack logical reasoning ability. So in order to solve the logical omniscience problem, we should find a way to distinguish relatively easy consequences from difficult ones. However, the expressivity of MEL is too poor for us to have the sufficient information we need to make the distinction. In MEL, and other alternative epistemic logics, only the content of knowledge is represented, and the temporal information of when knowledge is known is not indicated. But knowledge occurs in time, and we don't obtain all of our knowledge at once. Some is more difficult, so we need more time to deduce it, and some is trivial and can be had immediately. If we lack this temporal information of agent knowledge, in applications, we have to assume that agents possess all their knowledge at the same time, and accordingly know too much. Following this line of analysis, it is expected that the alternative epistemic logics proposed can validate inference rules of the following form:
For any $i \in \mathbb{N}$, there is a $j > i$ such that

$$(2) \qquad \frac{\vdash \phi \to \psi}{\vdash \mathbf{K}\phi^i \to \mathbf{K}\psi^j} \qquad \text{(Timed Knowledge Closure).}$$

The natural numbers i, j indicate the time instances when the agent obtains his knowledge of ϕ and ψ. The rule tells us that if the agent knows the premise of a logical implication at some time, then he knows the conclusion at a later time. The most important feature that these proposed logics should meet is that the time difference, $j - i$, should reflect in some way the computational difficulty of the implication.

With a rule of this kind we can determine what is known by the agent based on the temporal limitation of reasoning we set for the agent. In particular, we can argue what is the natural temporal elapse for a normal agent under normal circumstances to deduce his knowledge and then determine what log-

[7]Cf. [27] and [12] for details of these approaches, and others.

ical consequences will naturally be known by agents, given their premise knowledge. In this paper, we won't settle the argument, but if the temporal elapse of choice agrees with our intuition, this class of logical consequences should be the most favorable one. It won't be empty and won't include all logical consequences of the agent's knowledge. Furthermore, in a logic of this type where knowledge is temporal-dependent, we don't have to create a whole new epistemic model in order to determine what will be known by the agent with different temporal limitations. Thus in logics with this kind of rule, agents are not portrayed as logically omniscient. They do not know all the logical consequences of their knowledge within a reasonable time; however, they are capable of performing logical reasoning. Agents can continue to make efforts to know all logical consequences of their knowledge, especially if there are no temporal restrictions.

In this paper, we will present logics with the features we have described. Logics introduced in this paper will be modified from MEL, both semantically and syntactically. We will call these logics tMEL, timed Modal Epistemic Logic, to indicate that the moment of time when knowledge is known is made explicit. For each MEL logic, there will be a corresponding axiomatic-like deductive machinery which is supposed to be employed by an agent to increase his knowledge. When an agent will know a formula depends on how long it will take for the agent to deduce the formula by this machinery, and agents with different initial logical knowledge functioning as axioms in axiom systems have different logical strengths and will deduce different amounts of formulas. The framework of tMEL respects this diversity such that agents with different initial logical knowledge will be described by different logics and hence for each MEL logic, there is actually a collection of corresponding tMEL logics to be introduced.

Among all tMEL logics corresponding to the same MEL logic, there are always some which delineate agents with richer initial logical knowledge and have a close relationship with the MEL logic — theorems of MEL logic are exactly the theorems of these tMEL logics without considering the temporal components. This formal connection between MEL and tMEL is called the *realization theorem*. It shows that these MEL logics do have dynamic aspects themselves, but these aspects are only revealed in their tMEL counterparts. The implication of this connection theorem can be interpreted as follows. Knowledge acquisition happens in time, so a successful logic of knowledge must feature a temporal dimension of knowledge, implicitly or explicitly. Here the connection theorem justifies that MEL is truly a logic of the content of knowledge with a hidden temporal structure realized in tMEL. It rebuts the view that the problem of logical omniscience is inherent in possible world semantics, as many researchers suggest. The incapability of MEL, however, is the result of missing explicit temporal components from which model designers can determine what is known and what is not known by agents under various temporal restrictions, and then the success of applications of MEL will depend on how much temporal consideration should be taken into account in applications.

The Methodology and A Historical Note

The epistemic model we are going to present here is augmented from possible world semantics by adding a syntactical device to each world. Methodologically, this is similar to Fagin-Halpern's awareness approach ([11]), and the difference is that the syntactical device, also called an awareness function, we will introduce here is not simply a sieve to distinguish knowledge of different types, but is structured to record the time when the agent is aware of formulas through deductive reasoning. Fagin-Halpern once suggested the possibility of utilizing awareness functions such that time can be added. Our approach can also be regarded as a direct response to this suggestion.

This paper, however, originates from another tradition. Justification Logic begins with *Logic of Proofs* LP, introduced by Artemov in [1, 2], as an explicit proof counterpart of modal logic S4 with a provability reading suggested by Gödel [16]. New formula constructors called *proof terms* or *proof polynomials* t, and new atoms of formulas $t{:}\phi$ to mean t is a proof, evidence, or justification of ϕ are introduced. Today Justification Logic is a well-developed area of research. Many technical results and extensions have been studied.[8] tS4, the timed modal logic counterpart of S4, can be viewed as a numerical version of LP, and its axiom system is first introduced in [35] (in which $\mathrm{S4}^{\Delta}$ is the name for tS4) to study the relationship between axiomatic proofs of S4 and LP. A syntactical proof of the *realization theorem* between tS4 and S4 has been built there. Taking justificatory complexity as a measure of the computation efforts an agent should put on a derivation of a logical implication, Justification Logic can also be regarded as an epistemic logical framework without the assumption of logical omniscience. Under a carefully chosen limitation of justificatory complexity, we can define a natural class of logical consequences that the agent can produce. It is also shown in [35] that there exists an efficient translation between tS4 and LP that reflects an informal relation between justification and time. Justifications need time to be produced, and we gain knowledge through time by giving justifications. The study of tS4 and LP can complement each other. While LP has a more refined framework such that the reasoning history of knowledge can be traced, reasoning with the temporal feature of knowledge is more intuitive, and working with natural numbers and their linearity is easier than directly dealing with justification objects.

The first possible world and epistemic semantics for Justification Logic is given by Fitting in [15], which our semantics of tMEL basically follows, but modifications are made to have the semantics better fit our intuition concerning knowledge in the passage of time. In this paper, the focus will mainly be on tS4 logics, the MEL counterparts of S4, and the reason is only historical. S4 is the first MEL logic, and is the modal logic counterpart of the first Justification Logic. However, it will be clear from our construction that the techniques and methods for the study of S4 and tS4 can be uniformly extended to other MEL and tMEL logics. At the end of this paper we will discuss the semantic conditions for these logics, which will include logics with Axiom 5, the so-called axiom of Negative Introspection.[9]

2. Logics of tS4

To begin, we review MEL and possible world semantics, which will form the basis of tMEL semantics. The language of MEL is built up from a set of primitive propositions, Boolean connectives $\sim$ and $\to$ ($\wedge$ and $\vee$ are defined in terms of the other two), and a modal operator $\mathbf{K}$, together with parentheses for delimitation. In this paper, we only consider the case of one agent and discuss propositional knowledge. *Knowledge* or *epistemic* will be used broadly to cover some cases in which *belief* or *doxatic* might be a better term. Well-formed formulas are defined as usual; in particular, if ϕ is a formula, so is $(\mathbf{K}\phi)$, which means the agent knows ϕ, or ϕ is known. If there is no confusion will be made, the outmost parentheses are usually omitted. A *structure* or *a model* for MEL is a tuple $\langle W, R, \mathcal{V}\rangle$, where W is a set of worlds or epistemic alternatives, R is a binary relation defined on W, and $\mathcal{V}$ is a function assigning possible worlds to primitive propositions. The *satisfaction relation* in a structure M is recursively defined as follow:

$(M, w) \Vdash p$, where p is a primitive proposition iff $w \in \mathcal{V}(p)$,

[8] Cf. [6], [3], [4], [24], [15], [22], and [7], for instance.

[9] The Justification Logic counterpart of Axiom 5 is studied in [28] and [32].

$(M, w) \Vdash \sim\phi$ iff $(M, w) \nVdash \phi$,

$(M, w) \Vdash \phi \to \psi$ iff $(M, w) \nVdash \phi$ or $(M, w) \Vdash \psi$,

$(M, w) \Vdash \mathbf{K}\phi$ iff $(M, w') \Vdash \phi$ for all $w' \in W$ with wRw'.

We call a formula *valid in a structure* if it is satisfied in every world of the structure. Formulas which are valid in all structures compose the smallest MEL logic, *K*. For a more precise term, this is a logic of belief, not a logic of knowledge. Within this framework, to model agents under additional epistemic considerations, a subclass of structures are concerned. For example, in this paper we study the knowledge of an agent with *positive introspection*, which is characterized by the *reflexive* and *transitive* structure. Formulas valid in these structures form *S4*, the logic of knowledge proposed by Hintikka. It is not difficult to see that in this semantics if an implication $\phi \to \psi$ is valid in all structures, so is $\mathbf{K}\phi \to \mathbf{K}\psi$. The agent modeled in this semantics knows all the logical consequences of his knowledge.

2.1. Awareness by Deduction

The language of tS4, as well as that of the framework of tMEL in general, is similar to the language of MEL, except that natural numbers are also formula constructors and if ϕ is a tMEL formula, $(\mathbf{K}\phi^i)$, not simply $(\mathbf{K}\phi)$, is a tMEL formula. The natural numbers are used to model the passage of time. We only consider a simple structure of time: discrete, linear, and with an initial point. The intended meaning of $\mathbf{K}\phi^i$ is that the agent knows ϕ at time i, or ϕ is known at i. Writing the modal formula as $\mathbf{K}^i\phi$ is not recommended, as it implies the presupposition that there are different knowledge operators at different times. Notice, however, that the natural numbers are only assigned to formulas prefixed by knowledge operator $\mathbf{K}$. Other formulas are considered temporal invariances. In particular, the facts of the world captured by the primitive propositions are supposed to be unchanged in the course of the agent's reasoning.

An *awareness function* is a partial function that associates tMEL formulas with natural numbers. Given a formula ϕ and a natural number i, $\alpha(\phi){=}i$ means that the agent is aware of ϕ at time i and no earlier than i, i.e., the first time the agent is aware that ϕ is i. It is implicitly presupposed that the agent's awareness is monotone, that is, if the agent is aware of a formula at i, he is also aware of the formula at $i + 1$. The choice of α as a partial, not total, function is to manifest that it is not necessary that the agent be aware of all formulas.

Awareness functions are employed to represent the agent's deduction history by recording the time when a formula is derived. Each deduction history has to start from some base formulas which are not derived from any others. The agent might have them inherently, or hear them from someone else. Given a tuple $\mathcal{A} = \langle \mathbf{A}, f \rangle$, where $\mathbf{A}$ is a set of formulas and f is a total function assigning formulas in $\mathbf{A}$ to a number, we say that an awareness function α is *based on* $\mathcal{A}$, or $\mathcal{A}$ is a *base* of α, if it satisfies the following:

0. If $A \in \mathbf{A}$, then $\alpha(A) \le f(A)$. (*Initial Condition*)

We call $\mathbf{A}$ the *base set* of $\mathcal{A}$, and f the *base function* of $\mathcal{A}$. Sometimes we will simply write $\phi \in \mathcal{A}$ to mean $\phi \in \mathbf{A}$. We call α an $\mathcal{A}$-*awareness function*, or just write $\alpha_{\mathcal{A}}$, to indicate α is based on $\mathcal{A}$.

The most basic rule that we assume our agent can perform is *Modus Ponens* and for each step, he will use a unit of time to do it. We model these as ($\alpha(\phi)\!\downarrow$ means $\alpha(\phi)$ is defined):

1. If $\alpha(\phi \to \psi)\downarrow$ and $\alpha(\phi)\downarrow$, then
$\alpha(\psi) \le \max(\alpha(\phi \to \psi), \alpha(\phi)) + 1.$ (*Deduction by Modus Ponens*)

The reason we use *less-than or equal* ($\le$) and not simply *equal* ($=$) in the main clauses of the rules is that the agent might be aware of, say, ψ earlier in this case, since he derives it from other formulas.

Another general rule that we assume our agent can manipulate is that for base formulas, he is able to be aware that he knows them. Suppose, for example, someone at time i told the agent ϕ, and from this the agent is able to deduce "he knows ϕ at i." This deduction rule is formulated as:

2. If $A \in \mathbf{A}$ and $f(A) \le i$, then
$\alpha(\mathbf{K}A^i) \le i + 1.$ (*Deduction by $\mathcal{A}$-Epistemization*)

An $\mathcal{A}$-awareness function that satisfies the above conditions is called *normal*.

Finally, for an S4 (or tS4) agent with positive introspection, i.e., knowing what he knows, we assume he is able to be aware that he knows ϕ for any formula ϕ that he is aware of. This is modeled as follows:

3. For any ϕ, if $\alpha(\phi) \le i$, then
$\alpha(\mathbf{K}\phi^i) \le i + 1.$ (*Inner Positive Introspection*)

The word *inner* implies there will be an *outer* rule. We call the rule that the agent can perform here *inner* since it is the introspection about the awareness of a formula. The *outer* rule, which will be defined after we introduce the semantics, is introspection about the satisfaction of the knowledge of a formula.

It is also apparent that the *Inner Positive Introspection* condition is a general form of the condition *Deduction by $\mathcal{A}$-Epistemization*. We separate them here to demonstrate epistemically that the latter is more basic than the former (only formulas assumed in the base can be epistemized). The advantage of this separation will become clearer when we consider logics without Positive Introspection.

Given these conditions, we can build an awareness function which is purely affected by the formulas in the base. To some extent, this function is a complete characterization of our agent's reasoning ability. For two awareness functions α and β, we write $\beta \le \alpha$, if $\beta(\phi) \le \alpha(\phi)$ for any formula ϕ such that $\alpha(\phi)\downarrow$.

Lemma 2.1. Given an awareness base $\mathcal{A}$, there exists a unique (normal, S4, or none) $\mathcal{A}$-awareness function $\alpha^*_{\mathcal{A}}$, called *critical*, such that for any $\mathcal{A}$-awareness function α, $\alpha \le \alpha^*_{\mathcal{A}}$.

Informally speaking, the critical awareness function makes an agent aware of a formula as late as possible. The agent might be aware of the formula earlier only if additional information is possessed.

2.2. Semantics

We first consider the semantics for tMEL in general and then discuss the semantics for tS4 in particular. A *structure for tMEL* is a tuple $M = \langle W, R, \mathfrak{A}, \mathcal{V}\rangle$ where $\langle W, R, \mathcal{V}\rangle$ is a MEL structure and $\mathfrak{A}$ is a collection of awareness functions α_w indexed by the worlds $w \in W$. That is, for every world $w \in W$, there is one and only one $\alpha_w \in \mathfrak{A}$. The *satisfaction relation* of this semantics is defined as follows:

$(M, w) \Vdash p$, where p is a primitive proposition, iff $w \in \mathcal{V}(p)$;
$(M, w) \Vdash \sim\phi$ iff $(M, w) \nVdash \phi$;

$(M,w) \Vdash \phi \rightarrow \psi$ iff $(M,w) \nVdash \phi$ or $(M,w) \Vdash \psi$;
$(M,w) \Vdash \mathbf{K}\phi^i$ iff $(M,w') \Vdash \phi$ for all $w' \in W$ with wRw', and $\alpha_w(\phi) \leq i$.

We can easily recognize that only the last clause is altered from the standard MEL satisfication relation. It states that at time i, the agent knows ϕ in a world w if and only if the formula is true at all worlds accessible from w, and that the first time he is, by deduction or other means, aware of the formula is before or at i. Of course, setting the rule in this way, we presuppose that the agent won't forget what he knows. One reason for this setting is to simplify the argument; the other is that we consider the scenario in which the agent is goal-directed, directing all his efforts in logical reasoning, and hence in the course of the reasoning, it is assumed that he won't forget what he knows. Nevertheless, this presupposition is not essential to the construction of the framework; systems without the presupposition can be developed by adjusting the condition we set here.

Similar to MEL, agents are classified by subclasses of tMEL-structures. But subclasses now are also determined by the collections of awareness functions in the structures. We attribute properties to collections of awareness functions by their elements. For example, if a collection contains only normal awareness functions, we say the collection is *normal*, and if the collections contains only $\mathcal{A}$-awareness functions, we say the collection is an *$\mathcal{A}$-collection*, or is *based on $\mathcal{A}$*. For a tMEL-structure $M = \langle W, R, \mathfrak{A}, \mathcal{V} \rangle$, we say $\mathfrak{A}$ is *monotonic* if for any wRw', $\alpha_{w'} \leq \alpha_w$, that is, accessible worlds have the agent aware of formulas at an earlier time.

Definition 2.1. Given a base $\mathcal{A}$, we call a tMEL-structure $\langle W, R, \mathfrak{A}, \mathcal{V} \rangle$ a *tS4($\mathcal{A}$)-structure* if R is transitive and reflexive, and $\mathfrak{A}$ is normal, inner positive introspective, monotonic, and based on $\mathcal{A}$.

The idea behind this semantics should be clear and intuitive. In each world, we have the valuation function to tell us the truth or falsity of the outside world, and have the awareness function to tell us the dynamic mental behavior of the agent. In this way, the agent's varied epistemic abilities are also manifested by diverse relations between awareness functions in structures. For example, monotonicity says that the agent can only conceive of a world in which he himself might be aware of some formula at an earlier time because he might have some additional information in the conceived world, but he won't conceive of a world in which he himself is aware of fewer formulas since he, as an S4 agent, has strong evidence as to what he is aware of in this world.

We say a formula is *valid* in a tMEL-structure if the formula is satisfied at all worlds in the structure, and a formula is *tS4($\mathcal{A}$)-valid* if it is valid in all tS4($\mathcal{A}$)-structures. We denote this as $\vDash_{tS4(\mathcal{A})} \phi$, given that $\mathcal{A}$ is a base. Then the *logic of tS4($\mathcal{A}$)* is the set of tS4($\mathcal{A}$)-valid formulas.

Here are some properties of our model of knowledge and time, for an S4 agent. They are valid in all tS4-structures.

Classical tautologies;
$\mathbf{K}(\phi \rightarrow \psi)^i \rightarrow (\mathbf{K}\phi^j \rightarrow \mathbf{K}\psi^k)$ for $i, j < k$;
$\mathbf{K}A^i \rightarrow \mathbf{K}(\mathbf{K}A^i)^j \quad i < j$ if $A \in \mathcal{A}$;
$\mathbf{K}\phi^i \rightarrow \mathbf{K}\phi^j \quad i < j$;
$\mathbf{K}\phi^i \rightarrow \phi$;
$\mathbf{K}\phi^i \rightarrow \mathbf{K}(\mathbf{K}\phi^i)^j \quad i < j.$

The validity of almost of all these formulas directly follows the definitions of awareness functions, satisfication relations, and structures. We prove the last, which needs some care. Suppose $\mathbf{K}\phi^i$ is true at some world w and wRw'. Then, by the satisfication relation, $\alpha_w(\phi) \leq i$ and ϕ is true in w'. It follows that $\mathbf{K}\phi^i$ is true at every w' with wRw' because R is transitive and $\alpha_{w'}(\phi) \leq \alpha_w(\phi) \leq i$ ($\mathfrak{A}$ is monotonic). Since α_w is inner positive introspective, $\alpha(\mathbf{K}\phi^i) \leq i+1$. It follows that $\mathbf{K}(\mathbf{K}\phi^i)^j$ is true in w when $i < j$. Note that in the proof we need the conditions that the collection of awareness functions is monotonic and inner positive introspective.

For a tMEL-structure $\langle W, R, \mathfrak{A}, \mathcal{V}\rangle$, we say $\mathfrak{A}$ is *outer positive introspective* if all the awareness functions α_w in $\mathfrak{A}$ satisfy the following condition: if $(M, w) \Vdash \mathbf{K}\phi^i$, then $\alpha_w(\mathbf{K}\phi^i) \leq i+1$. It is an easy exercise to show that if $\mathfrak{A}$ is inner positive introspective, then it is outer positive introspective. The reason we introduce this outer rule is to make comparisons with the situation that occurs when we deal with the negative introspection condition. In that case, the outer rule implies the inner rule. We give a definition for the package of conditions on the collection of awareness functions that are related to positive introspection.

Definition 2.2. For a tMEL-structure $\langle W, R, \mathfrak{A}, \mathcal{V}\rangle$, we say $\mathfrak{A}$ is *positive regular* if it is normal, monotonic, and both inner and outer positive introspective.

Then a tMEL structure is a tS4($\mathcal{A}$)-structure if and only if R is reflexive and transitive, and $\mathfrak{A}$ is positive regular.

2.3. Logical Bases

We now turn our focus to the bases of awareness functions. The formulas in the base of an awareness function are the formulas of which we suppose the agent is intrinsically aware. We have not yet placed any restriction on the base in our definition. Therefore, bases can be composed of empirical facts, which are captured by primitive propositions, or of inconsistencies. But for now we are interested in bases which contains only logical truths, i.e., valid formulas. Later, tS4($\mathcal{A}$) logics with bases $\mathcal{A}$ of logical truths will be axiomatized.

To decide what could be counted as a logical base is not an easy task. Certainly, all formulas valid in all tS4 structures are logical truths, and with this definition we already have some interesting logical bases. For example, we can study the logic of an agent aware of all classical or intuitionistic tautologies; these logics by themselves are worth further study. However, we would like our definition to be more comprehensive. Consider the following case. Suppose the propositional tautology $\phi{=}A{\to}(B{\to}A)$ is the only element in a base $\mathcal{A}$, and then it can be shown that $\psi{=}\mathbf{K}A^i{\to}\mathbf{K}(B \to A)^j$ is a tS4($\mathcal{A}$)-valid formula for some $i < j$, but not valid in all tS4 structures. However, there seems no reason to exclude a base containing only formulas ϕ and ψ as a logical base. We will give a constructive definition of logical bases hinted at by this example.

Here is some terminology, most of which is standard. Given bases $\mathcal{A} = \langle \mathbf{A}, f\rangle$ and $\mathcal{B} = \langle \mathbf{B}, g\rangle$, $\mathcal{B} \subseteq \mathcal{A}$ means $\mathbf{B} \subseteq \mathbf{A}$ and $f(B) \leq g(B)$ for all $B \in \mathbf{B}$. We call a set of bases $\{\mathcal{A}_i(= \langle \mathbf{A}_i, f_i\rangle)\}_{i\in\mathbb{N}}$ an *ascending chain* if $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \ldots$, and we say a base $\mathcal{A}$ is the *limit* of the ascending chain if $\mathcal{A} = \bigcup \mathcal{A}_i$, i.e., $\mathbf{A} = \bigcup \mathbf{A}_i$ and $f(A) = \min\{f_i(A) : f_i(A){\downarrow}\}$.

We also say a base $\mathcal{A}$ is *(tS4-)sound over* a base $\mathcal{B}$ if for every $\phi \in \mathcal{A}$, $\vDash_{tS4(\mathcal{B})} \phi$, and we say $\mathcal{A}$ is *sound and completely over* $\mathcal{B}$ if $\mathcal{A}$ is sound over $\mathcal{B}$ and for every tS4($\mathcal{B}$)-valid formula ϕ, $\phi \in \mathcal{A}$.

Definition 2.3. To say that a base $\mathcal{A}$ is *tS4 logical* means at least one of following is true: (1) $\mathcal{A}$ is empty, (2) $\mathcal{A}$ is sound over a tS4 logical base, or (3) $\mathcal{A}$ is the limit of an ascending tS4 logical bases $\{\mathcal{A}_i\}_{i\in\mathbb{N}}$, where A_{i+1} is sound over A_i for each $i \in \mathbb{N}$.

If $\mathcal{B} \subseteq \mathcal{A}$, every tS4($\mathcal{B}$)-valid formula is a tS4($\mathcal{A}$)-valid formula. So it is not difficult to see that if $\mathcal{A}$ is a logical base, then for every $\phi \in \mathcal{A}$, $\vDash_{tS4(\mathcal{A})} \phi$ and hence $\vDash_{tS4(\mathcal{A})} \mathbf{K}\phi^i$ for $f(\phi) \leq i$, where f is the base function of $\mathcal{A}$. Also notice that the definition of logical bases is logically dependent. For different tMEL logics, there will be different logical bases to be concerned. We have the following lemma:

Lemma 2.2. Given a tS4 logical base $\mathcal{A}$, and a tMEL formula ϕ, if $\alpha^*_{\mathcal{A}}(\phi)$ is defined, then ϕ is tS4($\mathcal{A}$)-valid.

Logical bases also have a finitude feature. Call a logical base finite if its base set is finite.

Lemma 2.3. Given a tS4 logical base $\mathcal{A}$ and a tMEL formula ϕ, $\vDash_{tS4(\mathcal{A})} \phi$ if and only if there is a finite tS4 logical base $\mathcal{B} \subseteq \mathcal{A}$ such that $\vDash_{tS4(\mathcal{B})} \phi$.

This lemma can be proved semantically by first proving a compactness theorem. To save space, we leave it as a corollary of the completeness theorem, which we will show later when axiom systems are introduced.

3. More on Logical Bases

One advantage of the framework that we introduced above is its flexibility in modeling agents with different initial logical knowledge (logical bases). Given a tS4 logical base $\mathcal{A}$, we have the logic tS4($\mathcal{A}$) particularly describes the logical and temporal structure of the knowledge possessed by an agent with the logical strength determined by the logical base. In this section, we discuss several natural conditions on the logical bases, and some of these conditions, as we will show, turn out to determine the same class of logical bases.

The smallest logical base is the empty base, composed of the empty set and the empty function. It is clear and supported by the logic that an agent with empty base does not have any assured knowledge. No tS4($\emptyset$)-valid formula is of the form $\mathbf{K}\phi^i$. One thing should be clarified: given a logical base $\mathcal{A}$, the logic tS4($\mathcal{A}$) is not exclusively about the agent with base $\mathcal{A}$. Instead, it is a logic of an agent who has *at least* $\mathcal{A}$ as his base. Hence a formula that is tS4($\emptyset$)-valid is also tS4($\mathcal{A}$)-valid for every $\mathcal{A}$.

At the other extreme, there are logical bases in which every logical truth has been included. We call a tS4 logical base $\mathcal{A}$ *comprehensive* if for every tS4($\mathcal{A}$)-valid formula ϕ, $\phi \in \mathcal{A}$. Given an ascending chain of tS4 logical bases $\{\mathcal{A}_i\}_{i\in\mathbb{N}}$ where $\mathcal{A}_0$ is empty, and for every $i \in \mathbb{N}$, $\mathcal{A}_{i+1}$ is sound and completely over $\mathcal{A}_i$, then the limit of the ascending chain is comprehensive. This is a direct result following the finitude of logical bases. Since if $\vDash_{tS4(\mathcal{A})} \phi$, then, by Lemma 2.3, $\vDash_{tS4(\mathcal{A}_i)} \phi$ for some i, so $\phi \in \mathcal{A}$. Note that there is more than one comprehensive base. In the above construction, if the base functions of the logical bases in the chain are changed, we shall get different comprehensive logical bases. Among all these comprehensive bases, there is a maximal one. Call a base *principal* if its base function is the constant function 0. Let $\mathcal{A}$ be a tS4 logical base which is comprehensive and principal. Then it is not difficult to see that for any comprehensive tS4 logical base $\mathcal{B}$, $\mathcal{B} \subseteq \mathcal{A}$. We can also have the following result immediately: if $\vDash_{tS4(\mathcal{A})} \phi$, then $\vDash_{tS4(\mathcal{A})} \mathbf{K}\phi^i$ for any $i \in \mathbb{N}$.

Agents with comprehensive logical bases are unrealistic. They know too much from the beginning. So for realistic logical bases, some moderate conditions should be satisfied. Generally speaking, we would like the logical base to be smaller in size but without limiting the reasoning ability of the agent. One of these conditions is to demand that agents with logical bases of this kind be logically indistinguishable from agents with comprehensive bases. That is, given a tS4 logical base $\mathcal{A}$, the following is satisfied:

(i) There is a comprehensive base $\mathcal{B}$ such that $\vDash_{tS4(\mathcal{A})} \phi$ iff $\vDash_{tS4(\mathcal{B})} \phi$.

The other condition is from consideration of the awareness function. Since, in the semantics, the awareness function simulates the agent's deductive ability, we would like a logical base to be rich enough such that an agent with the base is able to be aware of every valid formula. This condition can be formulated in two ways:

(ii) if $\vDash_{tS4(\mathcal{A})} \phi$, then $\alpha^*_{\mathcal{A}}(\phi)\downarrow$, or

(ii') if $\vDash_{tS4(\mathcal{A})} \phi$, then $\{\alpha(\phi) \mid \alpha$ is a tS4 $\mathcal{A}$-awareness function and $\alpha(\phi)\downarrow\}$ is a finite set.

Condition (ii) is the converse of Lemma 2.2. It might be helpful to think of these conditions as follows: Lemma 2.2 says that if a base is logical, then it is sound, and our conditions here state that we would like the base to be complete, too. Of course, this way of speaking is nonstandard, since the agent's inference system is part of the semantics.

Finally, we might want our agent to have a logical base rich enough such that all the logical truths are known to him in every world in every structure. That is:

(iii) if $\vDash_{tS4(\mathcal{A})} \phi$, then $\vDash_{tS4(\mathcal{A})} \mathbf{K}\phi^i$, for some $i \in \mathbb{N}$.

Now these conditions come from different considerations; however, the interesting thing is that they determine the same class of tS4 logical bases. We will call a base *full* if it satisfies one of these conditions.

Theorem 3.1. The conditions (i), (ii), (ii'), and (iii) categorize the same class of logical bases.

Proof:
The equivalences between conditions (ii), (ii'), and (iii) are straightforward so we will prove the equivalence between conditions (i) and (ii). We first prove the direction from (i) to (ii). Given a tS4 logical base $\mathcal{A}$, suppose that $\vDash_{tS4(\mathcal{A})}\phi$ and that there is a comprehensive logical base $\mathcal{B}$ such that $\vDash_{tS4(\mathcal{A})}\psi$ iff $\vDash_{tS4(\mathcal{B})}\psi$ for each ψ. Then $\vDash_{tS4(\mathcal{B})}\phi$. Since $\mathcal{B}$ is comprehensive, so $\phi \in \mathcal{B}$ and hence $\vDash_{tS4(\mathcal{B})}\mathbf{K}\phi^i$ for $f(\phi) \leq i$ where f is the base function of $\mathcal{B}$. Now, following the assumption, there is some i such that $\vDash_{tS4(\mathcal{A})}\mathbf{K}\phi^i$ and hence $\alpha^*_{\mathcal{A}}(\phi)$ is defined. This completes the proof in one direction. For the other direction, suppose that for every tS4($\mathcal{A}$)-valid formula ϕ, $\alpha^*_{\mathcal{A}}(\phi)$ is defined. We define $\mathcal{B} = \langle \mathbf{B}, g\rangle$ with $\mathbf{B}=\{\phi \mid \vDash_{tS4(\mathcal{A})} \phi\}$ and $g(\phi)=\alpha^*_{\mathcal{A}}(\phi)$. Since for any $\phi \in \mathbf{B}$ and any tS4 $\mathcal{A}$-awareness function α, $\alpha(\phi) \leq g(\phi)(= \alpha^*_{\mathcal{A}}(\phi))$, α is a tS4 $\mathcal{B}$-awareness function. Hence every tS4($\mathcal{A}$)-structure is a tS4($\mathcal{B}$)-structure and every tS4($\mathcal{B}$)-valid formula is an tS4($\mathcal{A}$)-valid formula. By the definition of $\mathcal{B}$, every tS4($\mathcal{A}$)-valid formula is in $\mathcal{B}$. $\mathcal{B}$ is comprehensive. □

Logics with full bases will have the desirable property for epsitemic logic as we discussed in the introduction. The Rule of Timed Knowledge Closure is sound in these logics. Given a tS4 full logical base $\mathcal{A}$ and suppose $\vDash_{tS4(\mathcal{A})}\phi\to\psi$, then $\vDash_{tS4(\mathcal{A})}\mathbf{K}(\phi\to\psi)^k$ for some k, since $\mathcal{A}$ is full. From the construction of the semantics, especially the Rule of Deduction by Modus Ponens of awareness functions, it is not difficult to see that if the agent knows ϕ at some time i, then he is able to know ψ at a later time j.

Lemma 3.1. Suppose $\mathcal{A}$ is a full logical base, then the following rule holds:

$$\frac{\vDash_{tS4(\mathcal{A})} \phi \to \psi}{\vDash_{tS4(\mathcal{A})} \mathbf{K}\phi^i \to \mathbf{K}\psi^j} \text{ for some } j > i.$$

In the next section, after the axiom systems are introduced, we will see a concrete example of a full logical base.

4. Axiomatization

Infinitely many logics with their semantics have been introduced. We are able to axiomatize them in a uniform way. Given a tS4 logical base $\mathcal{A} = \langle \mathbf{A}, f \rangle$, a tS4($\mathcal{A}$) logic is introduced. Its sound and complete syntactical counterpart is the following:

Definition 4.1. tS4($\mathcal{A}$) Axiom Systems

Axioms	
A0 Classical propositional axiom schemes	
A1 $\mathbf{K}(\phi \to \psi)^i \to (\mathbf{K}\phi^j \to \mathbf{K}\psi^k)$ $\ i, j < k$	(Deduction by Modus Ponens)
A2 $\mathbf{K}A^i \to \mathbf{K}(\mathbf{K}A^i)^j$ $\ i < j$ if $A \in \mathbf{A}$ and $f(A) \leq i$	(Deduction by $\mathcal{A}$-Epistemization)
A3 $\mathbf{K}\phi^i \to \mathbf{K}\phi^j$ $\ i < j$	(Monotonicity)
A4 $\mathbf{K}\phi^i \to \mathbf{K}(\mathbf{K}\phi^i)^j$ $\ i < j$	(Positive Introspection)
A5 $\mathbf{K}\phi^i \to \phi$	(Truth Axiom)

Inference Rules	
R1 if $\vdash \phi \to \psi$ and $\vdash \phi$, then $\vdash \psi$,	(Modus Ponens)
R2 if $A \in \mathbf{A}$ and $f(A) \leq i$, then $\vdash \mathbf{K}A^i$	($\mathcal{A}$-Epistemization)

The validity of these axioms is easily established, and the soundness of the rule of Modus Ponens is evident. Since we only consider logical bases, the rule of $\mathcal{A}$-Epistemization is also justified. So soundness is proved. $\vdash_{tS4(\mathcal{A})} \phi$ is used to denote that ϕ is a theorem of the axiom system tS4($\mathcal{A}$).

Theorem 4.1. Given a tS4 logical base $\mathcal{A}$, $\vdash_{tS4(\mathcal{A})} \phi$ if and only if $\vDash_{tS4(\mathcal{A})} \phi$.

Proof:
We prove the completeness part of this theorem. We will construct a tS4($\mathcal{A}$)-structure composed of maximal $\mathcal{A}$-consistent sets. A set S of tMEL formulas is said to be $\mathcal{A}$-*consistent* if there is no finite subset $\{F_1, \ldots, F_n\}$ of S such that $\neg(F_1 \wedge \ldots \wedge F_n)$ is a tS4($\mathcal{A}$) theorem. The construction of a maximal such set is by the standard Lindenbaum construction. Let W be the set of all maximal $\mathcal{A}$-consistent sets and for any Γ, $\Gamma' \in W$, we define $\Gamma R \Gamma'$ if and only if $\Gamma^\sharp \subseteq \Gamma'$, where $\Gamma^\sharp = \{F \mid \mathbf{K}F^i \in \Gamma\}$, and define functions α_Γ and $\mathcal{V}$ by setting $\alpha_\Gamma(F) = \min\{i \mid \mathbf{K}F^i \in \Gamma\}$ and $\mathcal{V}(P) = \{\Gamma \mid P \in \Gamma\}$. We claim this $M = \langle W, R, \mathfrak{A}, \mathcal{V} \rangle$ with every $\alpha_\Gamma \in \mathfrak{A}$ is a tS4($\mathcal{A}$)-structure. The transitivity and reflexivity of R is implied by Positive Introspection and Truth Axiom. The $\mathcal{A}$-Epistemization rule implies that α_Γ is an $\mathcal{A}$-awareness function. It is not difficult to check that α_Γ also satisfies other conditions by applying these conditions' corresponding axioms. Finally, the collection of these awareness functions also satisfies

the monotonicity condition. Suppose $\alpha_\Gamma(F) = i$, $\mathbf{K}F^i \in \Gamma$. Since $\vdash_{tS4(\mathcal{A})} \mathbf{K}F^i \to \mathbf{K}(\mathbf{K}F^i)^j$, $\mathbf{K}(\mathbf{K}F^i)^j \in \Gamma$, so $\mathbf{K}F^i \in \Gamma'$ for any $\Gamma^\sharp \subseteq \Gamma'$. $\alpha_{\Gamma'}(F) \leq i$.

We now prove Truth Lemma: for every Γ, we have $F \in \Gamma$ if and only if $(M, \Gamma) \Vdash F$. The proof is by induction and most cases are trivial. We prove the modal case. If $(M, \Gamma) \Vdash \mathbf{K}F^i$, then $\alpha_\Gamma(F) \leq i$, so $\mathbf{K}F^i \in \Gamma$. For the other direction, if $\mathbf{K}F^i \in \Gamma$, $\alpha_\Gamma(F) \leq i$ and for any $\Gamma^\sharp \subseteq \Gamma'$, $F \in \Gamma'$, so by Induction Hypothesis, $(M, \Gamma') \Vdash F$, and hence $(M, \Gamma) \Vdash \mathbf{K}F^i$. This completes the proof of Truth Lemma. Now suppose ϕ is not provable in tS4($\mathcal{A}$), $\neg\phi$ is $\mathcal{A}$-consistent. $(M, \Gamma) \nVdash \phi$ with Γ a maximal $\mathcal{A}$-consistent set containing $\neg\phi$. ϕ is not tS4($\mathcal{A}$)-valid. □

When the awareness base has some special property, the description of the systems can be simplified. For example, when $\mathcal{A}$ is empty, the A2 axiom and R2 rule are void. When $\mathcal{A}$ is comprehensive, the clause "$A \in \mathcal{A}$" can be replaced by "$\vdash A$," and when the base is principal, "and $f(A) \leq i$" can be removed.

It is not difficult to recognize that this axiom system is almost completely parallel to the standard S4 axiom system, except that the language is richer and for each axiom, there are additional conditions on the temporal components. Each axiom describes one reasoning ability that the agent in discussion processes. Exceptions are Monotonicity (not to be confused with the monotonicity of collections of awareness functions) and Truth Axiom, which are descriptive properties of knowledge. The rule of $\mathcal{A}$-Epistemization is probably the most atypical. However, if the logical base $\mathcal{A}$ is comprehensive, then the rule directly corresponds to the Necessitation Rule of S4.

Interestingly, this type of axiom system itself gives us a logical base to consider, that is, the logical base which contains all these axioms. We will show below that it is full.

Lemma 4.1. Given a tS4 logical base $\mathcal{A}$, if every axiom instance belongs to $\mathcal{A}$, $\mathcal{A}$ is full.

Proof:
With the completeness and soundness results above, it is sufficient to prove that if ϕ is a theorem, then $\mathbf{K}\phi^i$ for some i is also a theorem. We prove the statement by induction on the length of the proof of ϕ. Suppose ϕ is an axiom. Then by $\mathcal{A}$-Epistemization, $\mathbf{K}\phi^i$ for $i \geq f(A)$ is a theorem. If ψ is derived from $\phi \to \psi$ and ϕ, then, by the Induction Hypothesis, both $\mathbf{K}(\phi \to \psi)^i$ and $\mathbf{K}\phi^j$ are theorems. Using axiom A1, we have a theorem $\mathbf{K}\psi^k$ for $k > i, j$. If $\mathbf{K}\phi^i$ is derived by $\mathcal{A}$-Epistemization, by applying axiom A2, $\mathbf{K}(\mathbf{K}F^i)^j$ for $j > i$ is a theorem. □

Definition 4.2. We say a tS4 logical base $\mathcal{A}$ is *axiomatically appropriate* if it contains all axiom instances of the schemes listed in the above system.

When $\mathcal{A}$ is axiomatically appropriate, the clause "$A \in \mathcal{A}$" can be replaced by "A is an axiom" in the axiom system.

Notice how axioms A1 and A2 are the only axioms used in the proof of Lemma 4.1 This result meshes with our intuition. Suppose that the agent we are going to investigate is aware of these axioms, that is, the agent's logical strengths are just like ours, and A1 and A2 explain that the agent can reason using Modus Ponens and $\mathcal{A}$-Epistemization. Then, with whatever formula we can prove ($\vdash \phi$), the agent can prove as well. So the agent is aware of ϕ at some time and hence the agent knows it ($\vdash \mathbf{K}\phi^i$, for some i).

These considerations explain why we list A2 separate from A4. When we consider logics without Positive Introspection, the A2 axiom plays a pivotal role in ensuring that the axiomatically appropriate bases of these logics are still full. For example, consider the axiom system tK, which consists of the tS4 axiom system without Positive Introspection and the Truth Axiom; tK is the tMEL counterpart of the minimal MEL K. Lemma 4.1 will still hold if we take $\mathcal{A}$ to be a tK logical base.

Realization Theorem

Our semantic work for tS4 is a modified possible world semantics, and it is clear that the tS4 axiom system displayed above is parallel to the standard axiom system S4. The question now is, what is the formal relation between tS4 and S4? As we have seen, tS4 is actually a collection of logics differentiated by their logical bases. Different logical bases exhibit different logical strengths of agents. So it can be expected that not every tS4 logic has a connection with S4. It is shown in [35] that the tS4 logic with a base which is principal and axiomatically appropriate can *realize* all S4 formulas.

Theorem 4.2. (The Realization Theorem)
Let $\mathcal{A}$ be a logical base that is principal and axiomatically appropriate. A MEL formula ϕ is an S4 theorem if and only if there is a corresponding tMEL formula ψ which is a tS4($\mathcal{A}$)-theorem such that ϕ is the resulting formula if we disregard all the time labels in ψ.

The original proof of the theorem is syntactical, and it is shown the realization procedure can be extended to MEL logics other than S4. A semantic proof of this theorem is a subject of future work. Experience suggests that fullness of the logical base should be a sufficient condition for tS4 realization of S4 theorems.

The Realization Theorem gives us a new insight into normal modal logics, especially from epistemic point of view. As we've argued, knowledge is accumulated over time, and MEL has its intuitiveness and proves useful in some cases, but has been considered as too idealized. Then the question is what the relation between MEL and time is. The theorem gives us an answer to this question. MEL knowledge does have temporal structure but the structure is only realized in tMEL. Each MEL theorem states the logical relation between the agent's known propositions, and when these known propositions are known is recorded in tMEL. Since in MEL the temporal information is missing, all the known propositions are regarded as known at the same time, it turns out that MEL is too idealized, that the agent pictured by MEL knows too much.

5. More Logics

In this section, we discuss how to extend the framework we introduced above to other MEL logics and their tMEL counterparts. It is well known that in MEL there is a corresponding relation between axiom schemes and conditions on the binary relations in structures. For every axiom system with a special axiom scheme, its sound and complete semantic counterpart will be the subclass of structures in which the binary relation satisfies the corresponding condition of the axiom scheme. The situation is similar for tMEL, but with some subtleties. The axiom schemes under consideration are not the schemes in MEL, but its tMEL counterparts. For the convenience of comparison, we will call the the axiom of Positive Introspection in the tS4 axiom systems *t4 Axiom*, and Truth Axiom *tT Axiom*. Corresponding semantic conditions of tMEL axiom schemes will be on both binary relations and collections of awareness functions. The nice thing is that the conditions on the binary relations of the tMEL axioms are the same as those conditions of the MEL counterparts of these tMEL axioms. So the binary relation of axiom tT is reflexive, and that of t4 is transitive. The basic condition on the collection of awareness functions is normal. No additional condition is needed for tT. But for t4, we need the collection to be positive regular. Below we list the needed conditions for logics combining these axioms. Let $M = \langle W, R, \mathfrak{A}, \mathcal{V} \rangle$ be a tMEL structure (notation note: tKT4 is the logic tK with additional axioms tT and t4, and other logics are named similarly):

	R	$\mathfrak{A}$
tK	no condition	normal
tKT	reflexive	normal
$tK4$	transitive	positive regular
$tKT4$ (S4)	transitive and reflexive	positive regular.

Now consider t5 Axiom: "$\neg \mathbf{K} F^i \to \mathbf{K}(\neg \mathbf{K} F^i)^j$ for $i < j$." Given a structure $M = \langle W, R, \mathfrak{A}, \mathcal{V} \rangle$, we say $\mathfrak{A}$ is *inner negative introspective* if for every $\alpha_w \in \mathfrak{A}$, $\alpha_w(\neg \mathbf{K} \phi^i) \leq i + 1$, provided $\alpha_w(\phi) \not\leq i$, say $\mathfrak{A}$ is *anti-monotonic* if for any $w \mathcal{R} w'$, $\alpha_w \leq \alpha_{w'}$, and say $\mathfrak{A}$ is *outer negative introspective* if for every $\alpha_w \in \mathfrak{A}$, $\alpha_w(\neg \mathbf{K} F^i) \leq i + 1$, provided $(M, w) \Vdash \neg \mathbf{K} F^i$; we then call $\mathfrak{A}$ *negative regular* if it is anti-monotonic and has both inner and outer negative introspection. For a tMEL logic with t5 axiom, its sound and complete semantics counterpart is the subclass of structures with *euclidean* binary relation and negative regular collection of awareness functions. It is easy to check that if $\mathfrak{A}$ is outer negative introspective, it is inner, and not the other way around.

Although theoretically we can come up with tMEL logics with t5 Axiom, from the epistemic point of view the axiom is rather dubious. The outer negative introspection rule says that the agent is aware of some formula in a world when some other formula is *satisfied* in the world. Unlike the inner rule in which the agent is aware of a formula because he is *aware* of another formula by his deduction and hence the rule can be considered as the agent reflects his reasoning process, the outer rule gives no hint as to what kind of reasoning process the agent is introspecting. There might be one such process (by which the agent learns that $\neg \mathbf{K} F^i$ is satisfied in the world), but it is not in the model. Then a structure in which the collection of awareness functions is negative regular can also describe an agent who happens to be systematically aware of some formulas which are true at a world. Further work needs to be done to make t5 case.

6. Conclusion

It probably isn't too much of an exaggeration to say that the logical omniscience problem is the most important threat to the enterprise of epistemic logic. From the beginning, philosophers have questioned the possibility of epistemic logic through this problem.[10] Later, epistemic logic finds its applications in many practical studies, but it is always argued that the knowledge modeled in epistemic logic is too idealized. The most successful applications of epistemic logic is in the study of distributed systems. One reason for this success is that the logical omniscience problem is not a problem in the application. The processes in systems don't produce their own knowledge. It is up to us, the model designers, to ascribe external knowledge to these processes for the study of the structures and behaviors of the systems.

[10] See [20]. Hintikka in [19] also mentions that Chomsky makes the same point in [9].

However, when we apply epistemic logic to intelligent agents, things are different. They are supposed to produce their own knowledge, and none of them can create as much knowledge as suggested by the standard modal epistemic logic. These seem to provide additional evidence that what matters is the computational efforts that the agent should make to produce knowledge, and that the temporal dimension of knowledge can help to reveal the information.

In this paper, we introduced a general epistemic framework tMEL to deal with the problem of logical omniscience. tMEL is the explicit temporal counterpart of MEL. Its semantics is augmented from the standard possible world semantics such that each world is equipped with an awareness function to tell us how the agent deduces his knowledge over time, besides a valuation function assigning the primitive truths of the world. With the formalism of tMEL we can express some meaningful epistemic statements which can't be stated in MEL. For example, the monotonicity of knowledge can be expressed as $\mathbf{K}\phi^i \rightarrow \mathbf{K}\phi^j$ for $i < j$. Also tMEL sustain a great range of logics, each of which reasons about the epistemic aspects of agents with different reasoning strengths determined by their initial logical knowledge. These diverse logics, even disregarding their temporal structures, are worthy of their own study.

Some unrealistic assumptions are still made in our framework. For example, we assume that agents are able to make all the deduction steps that they haven't made within a single unit of time, and suggest that an S4 or a tS4 agent is able to know that they know a formula in one step of positive introspection. Hence adjustments on the framework might be needed according to practical considerations. We also assume that agents employ axiomatic-like reasoning systems to deduce their knowledge, but with respect to the human capacity, a natural deduction type of reasoning might be preferable. However, the primary purpose of this paper is to make the point that knowledge is dynamic in nature and without this dynamic feature revealed, it is impossible to have an epistemic logical system free from the problem of logical omniscience. Our introduction of tMEL logics just makes a case that an epistemic framework with the dynamic feature of knowledge manifested can be developed. Once the temporal dimension of knowledge is made explicit, we can readily distinguish what is easily to know from what difficult, and without this distinction, the solutions proposed so far either do not solve the problem of logical omniscience, or do not model agent with full logical reasoning ability. So if the logic omniscience problem is taken seriously, we are suggesting the right direction for the future studies in epistemic logic.

Acknowledgements

I would like to thank Profefossor S. Artemov for encouraging this work, and especially thank an anonymous reviewer for many practical suggestions to improve this paper. Special thanks to Karen Kletter for editing and proofreading of this article.

References

[1] Artemov, S. N.: *Operational modal logic*, Technical Report MSI 95-29, Cornell University, 1995.

[2] Artemov, S. N.: Explicit provability and constructive semantics, *Bulletin of Symbolic logic*, **7**(1), 2001, 1–36.

[3] Artemov, S. N.: Justified common knowledge, *Theoretical Computer Science*, **357**(1), 2006, 4–22.

[4] Artemov, S. N.: The Logic of Justification, *The Review of Symbolic Logic*, **1**(4), December 2008, 477–513.

[5] Artemov, S. N., Kuznets, R.: Logical Omniscience Via Proof Complexity, in: *Computer Science Logic, 20th International Workshop, CSL 2006, 15th Annual Conference of the EACSL, Szeged, Hungary, September 25–29, 2006, Proceedings* (Z. Ésik, Ed.), vol. 4207 of *Lecture Notes in Computer Science*, Springer, 2006, 135–149.

[6] Artemov, S. N., Nogina, E.: Introducing Justification into Epistemic Logic, *Journal of Logic and Computation*, **15**(6), 2005, 1059–1073.

[7] Brezhnev, V., Kuznets, R.: Making knowledge explicit: How hard it is, *Theoretical Computer Science*, **357**, 2006, 23–34.

[8] Chellas, B. F.: *Modal logic*, Cambridge University Press, 1980.

[9] Chomsky, N.: *On the Generative enterprise: a discussion with R. Huybregts and H. van Riemsdijk*, Dordrecht/Cinnaminson: Foris, 1982.

[10] Eberle, R. A.: A logic of believing, knowing, and inferring, *Synthese*, **26**(3), 1974, 356–382.

[11] Fagin, R., Halpern, J. Y.: Belief, awareness, and limited reasoning, *Artificial Intelligence*, **34**, 1988, 39–76.

[12] Fagin, R., Halpern, J. Y., Moses, Y., Vardi, M. Y.: *Reasoning about knowledge*, MIT Press, 1995.

[13] Feferman, S., Ed.: *Kurt Gödel Collect Works*, Oxford, 1986-2003, 5 volumes.

[14] Fitting, M. C.: A logic of explicit knowledge, *In L. Behounek and M. Bilkova (Eds.), Logica Yearbook 2004*, Filosophia, 2005.

[15] Fitting, M. C.: The logic of proofs, semantically, *Annals of Pure and Applied Logic*, **132**, 2005, 1–25.

[16] Gödel, K.: Vortrag bei Zisel, Translated as *Lecture at Zisel's* in [13] III, 62-113, 1938.

[17] Halpern, J. Y., Pucella, R.: Dealing with logical omniscience, *Proceedings of the 11th conference on Theoretical aspects of rationality and knowledge*, 2007.

[18] Hintikka, J.: *Knowledge and Belief: An Introduction to the Logic of the Two Notions*, Cornell University Press, 1962.

[19] Hintikka, J.: Reasoning about knowledge in philosophy: the paradigm of epistemic logic, *Proceedings of the 1986 Conference on Theoretical aspects of reasoning about knowledge*, Morgan Kaufmann Publishers Inc., Monterey, California, United States, 1986.

[20] Hocutt, M.: Is Epistemic Logic Possible?, *Notre Dame Journal of Formal Logic*, **13**, 1972, 433–453.

[21] Konolige, K.: *A Deduction Model of Belief*, Morgan Kaufmann, San Francisco, 1986.

[22] Krupski, N. V.: On the complexity of the reflected logic of proofs, *Theoretical Computer Science*, **357**(1), 2006, 136–142.

[23] Levesque, H.: A logic of implicit and explicit belief, *Proceedings AAAI-84*, Austin, TX, 1984.

[24] Mkrtychev, A.: Models for the Logic of Proofs, *LFCS* (S. I. Adian, A. Nerode, Eds.), 1234, Springer, 1997.

[25] Montague, R.: Universal Grammar, *Theoria*, **36**, 1970, 373–98.

[26] Moore, R. C., Hendrix, G.: *Computational models of beliefs and the semantics of belief sentences*, Technical Report Technical Note 187, SRI International, 1979.

[27] Moreno, A.: Avoiding Logical omniscience and perfect reasoning: A survey, *AI Communications*, **11**(2), 1998, 101–122.

[28] Pacuit, E.: A note on some explicit modal logics, *Proceedings of the Fifth Panhellenic Logic Symposium*, 2005.

[29] Parikh, R.: Logical Omniscience, *LCC '94: Selected Papers from the International Workshop on Logical and Computational Complexity*, Springer-Verlag, London, UK, 1995.

[30] Parikh, R.: Sentences, belief and logical omniscience, or what does deduction tell us?, *The Review of Symbolic Logic*, **1**(04), 2009, 459–476.

[31] Rantala, V.: Impossible worlds semantics and logical omniscience, *Acta Philosophica Fennica*, **35**, 1982, 18–24.

[32] Rubtsova, N.: Evidence Reconstruction of Epistemic Modal Logic S5, *CSR* (D. Grigoriev, J. Harrison, E. A. Hirsch, Eds.), 3967, Springer, 2006.

[33] Scott, D.: Advice in modal logic, in: *Philosophical Problems in Logic* (K. Lambert, Ed.), Reidel, 1970.

[34] Vardi, M. Y.: On epistemic logic and logical omniscience, *Proceedings of the 1986 Conference on Theoretical aspects of reasoning about knowledge*, 1986.

[35] Wang, R.-J.: *On Proof Realization On Modal logic*, Technical Report TR-2009003, CUNY Ph.D. Program in Computer Science, 2009.

[36] von Wright, G. H.: *An Essay in Modal Logic*, North Holland Publishing Co., Amsterdam, 1951.