Non-circular Proofs and Proof Realization in Modal Logic

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Abstract

In this paper a complete proper subclass of Hilbert-style S4 proofs, named non-circular, will be determined. This study originates from an investigation into the formal connection between S4, as Logic of Provability and Logic of Knowledge, and Artemov's innovative Logic of Proofs, LP, which later developed into Logic of Justification. The main result concerning the formal connection is the *realization theorem*, which states that S4 theorems are precisely the formulas which can be converted to LP theorems with proper justificational objects substituting for modal knowledge operators. We extend this result by showing that on the proof level, non-circular proofs are exactly the class of S4 proofs which can be realized to LP proofs. In turn, this study provides an alternative algorithm to achieve the realization theorem, and, a logical system, called $S4^{\Delta}$, is introduced, which, under an adequate interpretation, is worth studying for its own sake.

1 Introduction

One of many applications of modal logic in computer science is to serve as logic of knowledge, for reasoning about the information transmissions in distributed systems (e.g. [16, 8, 14]), or about the intentional level of multiagent systems in general ([7, 22]). Artemov's Logic of Proofs, LP ([1, 2]), later developing into Justification Logic ([10, 9, 3, 4]), enhances the expressivity of modal epistemic logic by introducing justification into the language. Formulas of the like $t{:}F$ are introduced with the intended meaning that "*t is a proof of F*" or "*t is a justification of F*, " where t is a structural object, called *proof term* or *proof polynomial*, to stand for an explicit proof in formal arithmetic, or a justificational object. One of the main theorems concerning LP is about its formal relation with the modal logic S4. The *realization theorem* says that S4 theorems are exactly the formulas which can be turned into LP theorems by substituting suitable proof terms for the modal occurrences. Interpreted epistemically, the theorem shows that there is indeed a justification structure embedded in S4, as logic of knowledge, which can only be explicitly disclosed in the formalism of LP. The realization theorem is also the motivation for the introduction of Logic of Proofs. As a long standing question concerning the arithmetic foundation of intuitionistic logic, Gödel took the first step to embed intuitionistic logic into S4, as logic of provability ([12]), and Artemov furnished LP with a formal arithmetic semantics and then showed the realization theorem to complete the project.

Accordingly, a constructive syntactical proof for the realization theorem is offering an algorithmic procedure to extract the reasoning processes, the justification objects, from the logic of knowledge S4, and hence worth further attention. However, we find it is interesting and also puzzling in the original procedure given in [1, 2] and later improved in [5],[1] that it makes a detour to analyze cut-free Gentzen-style S4 proofs, even though originally LP is introduced in Hilbert-style and presented in a way that it is almost a realized counterpart of the standard Hilbert-style S4 system; and proof terms, which are also suggested to be regarded as combinators in some general way ([2]), are best understood as encoding proofs in Hilbert-style. So naturally, questions are raised: What happens to the Hilbert-style S4 proofs? What is the formal relation between S4 proofs and LP proofs, if both in the style of Hilbert? Can we extend the result of the realization theorem to concern S4 proofs, instead just of theorems? Thus although the realization theorem is introduced with its importance in application, it seems to suggest a deeper insight of the proof structure of modal logic. One of the contributions of this paper is just to determine a complete proper subclass of Hilbert-style proofs of S4, called *non-circular*, and show that this is exactly the class of proofs which can be realized to LP proofs.

In the present paper, we will first give a characterization of non-circular S4 proofs and then endeavor to show that the class is complete in the sense that every S4 theorem has a non-circular proof. It is our long-term goal to find an algorithm that can directly turn circular proofs into non-circular, but, partly because a proof-theoretical tool like cut elimination and normal-

[1] There is a semantic proof for the realization theorem based on a possible-world style of semantics for LP. See [10]. A recent constructive realization algorithm is developed based on cut-free proof systems of nested sequents in [13].

ization which can generates normal form of Hilbert-style proofs does not exist yet, in the following we will adopt an alternative procedure. For, as we know, there is a natural way of translating proofs in Gentzen-style to Hilbert-style, we will show that, following the translation, the Hilbert-style proofs obtained from cut-free proofs are non-circular. This result, which in turn justifies the completeness of non-circular proofs, explain in a way why the detour in the original proof of the realization theorem takes place and why the proof works well.

In the course of the discussions, a novel logical system called $S4^{\Delta}$ will be introduced to accomplish the above result. $S4^{\Delta}$ has numerical labels for each modal occurrence, which are designed to detect the non-circularity of S4 proofs. We will show that non-circular S4 proofs are precisely those that can turn into $S4^{\Delta}$ proofs by getting suitable numerical labels, and then prove that there is an efficient algorithm which can turn $S4^{\Delta}$ proofs into LP proofs, and vice versa. Putting all these ingredients together, we have the main result concerning the proof realization procedure connecting non-circular S4 proofs and LP proofs, and then the overall process also provides an alternative algorithm for the realization between theorems of S4 and LP.

$S4^{\Delta}$ as demonstrated here is an immediate logic between S4 and LP. It should serve well as a technical tool for the future study on the proof-theoretical structures of the logics on the both sides. But it is also an interesting logic worth studying for its own sake, depending on the interpretation of the numerical labels in the language of $S4^{\Delta}$. For one, with a direct connection with the interpretation of proof terms in LP as standing for explicit proofs, these labels could be understood as the lengths of proofs, and then $S4^{\Delta}$ is taken to be a logical system for studying an important syntactical property of proofs. And for another, these labels can be understood as representing the time that the modeled reasoner takes to perform the inferences to obtain knowledge, and hence $S4^{\Delta}$ can be regarded as a logical formalism for reasoning about knowledge and the time of reasoning. Once we understand $S4^{\Delta}$ in this way, our work will have fruitful consequences with an epistemic interpretation.[2]

[2] There are preliminary version of this pape in [19, 21]. Following that, the logic of knowledge and time has been developed [18, 20], in which $S4^{\Delta}$ is renamed as tS4 as one of the timed Modal Epistemic Logics.

2 The Systems

We begin with an introduction of the starting point of this project, that is, to establish a proof realization procedure between systems S4 and LP. S4 is a normal modal logic with language $L_\Box$ built up from propositional letters $\mathcal{P}$, boolean connectives $\neg, \vee, \wedge, \rightarrow$, and an unary modal operator $\Box$. The standard S4 (Hilbert-style) proof system is the following:

Axiom Schemes:

A0 axiom schemes of classical propositional logic
A1 $\Box(F \rightarrow G) \rightarrow (\Box F \rightarrow \Box G)$
A2 $\Box F \rightarrow \Box(\Box F)$
A3 $\Box F \rightarrow F$

Inference rules

R1 $F, F \rightarrow G \vdash G$ "modus ponens"
R2 $\vdash \Box F$, if $\vdash F$ "necessitation"

On the other side, LP can be viewed as a multimodal logic with *proof terms*, *Tm*, as modalities. Proof terms are built up from *proof constants* $\mathcal{C}$, *proof variables* $\mathcal{X}$, and basic proof operations: *application* $\cdot$, *proof checker* ! and *indeterminate choice* $+$. With $c \in \mathcal{C}$ and $x \in \mathcal{X}$, the grammar for the proof terms is: $t{:}{=}c|x|t{\cdot}t|t{+}t|!t$, and if ϕ is a formula in the language of LP, so is $t{:}\phi$. The system of LP introduced in [2] is the following:

Axiom Schemes:

A0 axiom schemes of classical propositional logic
A1 $s{:}(F \rightarrow G) \rightarrow (t{:}F \rightarrow (s{\cdot}t){:}G)$
A2 $s{:}F \rightarrow !s{:}(s{:}F)$
A3 $s{:}F \rightarrow F$
A4 $s{:}F \rightarrow (s{+}t){:}F$, $s{:}F \rightarrow (t{+}s){:}F$

Inference rules:

R1 $F, F \rightarrow G \vdash G$ "modus ponens"
R2 $\vdash c{:}F$ for $c \in \mathcal{C}$, if $\vdash F$ and F is an axiom
"axiom necessitation"

Notice that in these systems, there is flexibility in the choice of the axiom scheme A0. Any complete classical propositional axiom schemes can be wrapped together to be A0. For the purpose of this paper, we will assume that all the systems discussed here employ the same A0 axiom scheme, and we will call the axiom schemes other than A0 *modal axiom schemes.*

At this point we give a rough definition of *realization*, which will be formally formulated later. Of a modal formula in $L_\Box$, an LP-style formula is a realization, or sometimes called an explicit counterpart since the justifications of knowledge statements are explicitly stated, provided the formula is obtained from substituting proof terms for the modal occurrences in the modal formula. So we can see that the systems S4 and LP with only few exceptions are parallel to each other: every axiom scheme and rule in S4 has an explicit counterpart in the axiom system of LP, and every axiom scheme and rule in LP is an explicit counterpart of some axiom scheme and rule in S4. In addition, we can actually remove the exceptions by introducing both variants of S4 and LP. We will denote the system with the rule of *axiom necessitation* "$\vdash \Box F$, if $\vdash F$, and F is an axiom" substituting for the R2 rule of *necessitation* in S4 as S4$'$ and the system with axiom scheme A4 "$\Box F \to \Box F$" adding to S4$'$ as S4$''$. For LP, we introduce the following variant ELP (Below both $o(s)$ and $\dot{o}(s)$ denote proof terms of the form of finite sum with s as its summand (e.g., $t_1{+}s{+}t_2$), and $o(s){=}s$ is possible, whereas $\dot{o}(s){=}s$ is not):

Axiom Schemes:

- A0 axiom schemes of classical propositional logic
- A1 $s{:}(F \to G) \to (t{:}F \to o(s{\cdot}t){:}G)$
- A2 $s{:}F \to o(!s){:}(s{:}F)$
- A3 $s{:}F \to F$
- A4 $s{:}F \to \dot{o}(s){:}F$

Inference rules:

- R1 $F, F \to G \vdash G$ "modus ponens"
- R2 $\vdash o(c){:}F$ for $c \in \mathcal{C}$, if $\vdash F$ and F is an axiom "axiom necessitation"

The system with the axiom scheme A4 "$s{:}F \to \dot{o}(s){:}F$" removed from ELP is called ELP$^-$, and the system with the *necessitation* rule "$\vdash o(c){:}F$ for $c \in \mathcal{C}$, if $\vdash F$" substituting for the R2 rule of *axiom necessitation* in ELP$^-$ is called GELP$^-$. Systems S4, S4$'$ and S4$''$ prove the same set of theorems, but systems LP, ELP, ELP$^-$ and GELP$^-$ do not. However, every S4 theorem, and hence every S4$'$ and S4$''$ theorem, can be realized to a theorem in these systems of LP variants, and there would be syntactical translations between proof terms such that theorems in one of these systems can be translated into theorems in another one. The reason that we introduce the system ELP, and the notations $o(s)$ and $\dot{o}(s)$ will be clear when a proof realization procedure is formally discussed.

Now we can see there are complete parallelisms between systems S4 and GELP$^-$, between S4$'$ and ELP$^-$, and between S4$''$ and ELP, and it is natural to expect that these parallelisms can be extended to between proofs. We expect there is a line-to-line proof realization procedure, by which we mean a procedure to establish a set of proof terms such that a proof in S4, S4$'$ or S4$''$, can be turned into a proof in GELP$^-$, ELP$^-$, or ELP, respectively, and, furthermore, an axiom is turned into its corresponding explicit axiom, and a formula derived by a rule into a formula derived by its corresponding explicit rule.

However, this is just not the case. As it is shown in the following example, not all proofs in the S4-systems have this kind of straightforward line-to-line proof realization. This also shows that the task of establishing a procedure that can realize S4 theorems to LP theorems is not as trivial as we might think at our first pass of these systems. Here's a fragment of a proof:

$\phi_1 \equiv \Box(P \to Q) \to (\Box P \to \Box Q)$
$\phi_2 \equiv \Box(Q \to P) \to (\Box Q \to \Box P)$
$\phi_3 \equiv \Box(P \to Q) \to (\Box(Q \to P) \to (\Box P \to \Box P))$
$\phi_4 \equiv \Box(P \to Q) \to (\Box(Q \to P) \to (\Box Q \to \Box Q))$
$\phi_5 \equiv \Box(P \to Q) \to (\Box(Q \to P) \to ((\Box P \to \Box P) \wedge (\Box Q \to \Box Q)))$.

To save space, the intended proof is not listed in full here. Some steps are skipped, but the proof that we have in mind is that ϕ_3 is derived from ϕ_1 and ϕ_2 through some kind of general classical syllogism such that the subformula $\Box Q$ is removed. Notice that in the realization of this part of the proof the two occurrences of $\Box Q$ must be realized to the same LP formula in order that the classical syllogism can be parallelly applied in the corresponding LP-systems. Similarly, ϕ_4 is intended to be derived from ϕ_1 and ϕ_2 by removing $\Box P$, and finally ϕ_5 is derived from ϕ_3 and ϕ_4 through classical propositional logic.

Given that ϕ_1 and ϕ_2 are instances of modal axioms, their realizations should be explicit axioms:

$\phi_1^r \equiv s{:}(P \to Q) \to t{:}P \to o(s \cdot t){:}Q$
$\phi_2^r \equiv u{:}(Q \to P) \to v{:}Q \to o(u \cdot v){:}P$

for some proof terms s, t, u, v, and since syllogisms are applied, $v{=}o(s \cdot t)$ and $t{=}o(u \cdot v)$ must be the cases. Now due to the complexity of proof terms, no solution to these equations is possible, and hence the naive proof realization procedure doesn't work for this instance.

In the remaining of this paper, we will investigate the type of proofs just exemplified by this instance. The formal definition of non-circularity of proofs will be given in the next section. The complication of the section is to provide a machinery for reasoning the formula occurrences in a Hilbert-style proof. This new machinery should be handy for the future work on analysis of Hilbert-style proofs in general. The completeness of non-circular proofs will be established in Section 4. We will introduce logical systems S4$^{\Delta}$

and its variants and analyse cut-free Gentzen-style S4 proofs to accomplish the task. In Section 5, a direct algorithm which turns an $S4^{\Delta}$ proof, and therefore a non-circular S4 proof, into an LP proof, will be given. This result justified our earlier claim that non-circular S4 proofs are exactly the class of proofs that can be realized into LP proofs. Also, this algorithm together with procedures introduced in the previous sections provides an alternative algorithm for finding suitable proof terms to convert S4 theorems into LP theorems.

3 Non-Circular Proofs

Our following discussions of Hilbert-style proofs will be based on the analysis of formula occurrences, and hence a formal definition of occurrences and formulas at occurrences will be helpful and make clear our arguments. In a way, we will give an abstract description of paths of the parse tree of a formula ϕ to denote the positions, *occurrences*, of its subformulas, and the function $\phi(x)$ to denote the subformula at the occurrence x. The language of occurrences $\mathcal{O}$ are sequences of letters a, b, and $\star$. The symbol . (dot) doesn't belong to the formal syntax but will sometimes be written within a sequence to increase readability. We use a, b to denote the left and right positions of a binary operator, and $\star$ to denote the position of the operand of an unary operator. $\circ$ denotes a metavariable for a connective.

Definition 3.1 (occurrence).
Let $\phi \in L_{\Box}$. We simultaneously define the set of occurrences in ϕ, $\mathcal{O}(\phi)$, and the function $\phi(\cdot)$ which maps an occurrence in ϕ to the subformula of ϕ at the occurrence. Let ϵ be the empty sequence.

1. *$\epsilon \in \mathcal{O}(\phi)$ and $\phi(\epsilon) = \phi$,*
2. *If $x \in \mathcal{O}(\phi)$ and $\phi(x) = (\psi \circ \theta)$, then $x.a, x.b \in \mathcal{O}(\phi)$ and $\phi(x.a) = \psi$, and $\phi(x.b) = \theta$,*
3. *if $x \in \mathcal{O}(\phi)$ and $\phi(x) = (\circ\psi)$, then $x.\star \in \mathcal{O}(\phi)$ and $\phi(x.\star) = \psi$,*
4. *Furthermore, we extend the definition on formulas to sequences of formulas, such as proofs. Let $\mathcal{D}$ be a sequence of n formulas in $L_{\Box}$ and ϕ be the k-th element of $\mathcal{D}$, then $i \in \mathcal{O}(\mathcal{D})$ for any $i \leq n$ and $\mathcal{D}(k) = \phi$.*

Thus suppose $(A \rightarrow \Box B) \rightarrow A$ is the second element of a proof $\mathcal{D}$, then $2ab.\star \in \mathcal{O}(\mathcal{D})$ and $\mathcal{D}(2ab.\star) = B$. Here are some facts about the notion of occurrence ($x, y, z \in \mathcal{O}$):

- $\epsilon.x \equiv x \equiv x.\epsilon$,

- if $x \in \mathcal{O}(\phi)$ and $y \in \mathcal{O}(\phi(x))$, then $x.y \in \mathcal{O}(\phi)$, and $\phi(x.y) = \phi(x)(y)$,
- if $\phi(x) = \phi(y)$, then $x.z \in \mathcal{O}(\phi)$ iff $y.z \in \mathcal{O}(\phi)$,
- if ρ is a propositional letter substitution, and ϕ^ρ is the result of the substitution, then $\mathcal{O}(\phi) \subseteq \mathcal{O}(\phi^\rho)$.

A Hilbert-style proof is normally defined as a sequence of formulas which are either axiom instances or derived by rule applications. However, with the machinery of formula occurrences it is possible to define a Hilbert proof as a sequence of formulas where subformulas at some of the occurrences are mandatory to be equal, in order for a formula in the sequence to be an axiom or derived by a rule. In the following, the definition of *proof equivalence relation* is kind of trying to capture this idea. We call formulas of the form $\Box\phi$ *m-formulas*, and $m(\mathcal{D})$ is used to denote the set of occurrences of m-subformulas in $\mathcal{D}$. In this paper, we only care about the relation between m-subformula occurrences in a proof. We will use a *label function* to supply labels for m-formula occurrences in order to relate them. An m-formula $\Box P$ is atomic if $P \in \mathcal{P}$. Two atomic m-subformula occurrences are related to each other if they get the same label, and two general subformula occurrences are related to each other if all their m-subformula occurrences get the same labels. Here are the formal definitions.

Definition 3.2 (proof equivalence relation).
Let $\mathcal{D}$ be a proof in S4, S4′ or S4″, and k a natural number. An equivalence relation $\sim$ on $\mathcal{O}(\mathcal{D})$ is called a proof equivalence relation *if it satisfies:*

1. *if $\mathcal{D}(k){=}A^\rho$ with A an axiom scheme, ρ a propositional letter substitution, and $A(x){=}A(y){\in}\mathcal{P}$ for $x, y{\in}\mathcal{O}(A)$, then $k.x{\sim}k.y$,*[3]
2. *if $\mathcal{D}(k)$ is a substitutional instance of the axiom scheme A2, i.e. $\mathcal{D}(k){=}\Box\phi{\to}\Box(\Box\phi)$, then furthermore $k.a{\sim}k.b\star$,*
3. *if $\mathcal{D}(k){=}\psi$ is derived from $\mathcal{D}(i){=}\phi$ and $\mathcal{D}(j){=}\phi{\to}\psi$ by modus ponens, then $k{\sim}j.b$ and $i{\sim}j.a$,*
4. *if $\mathcal{D}(k){=}\Box\phi$ is derived from $\mathcal{D}(i){=}\phi$ by necessitation (in S4) or axiom necessitation (in S4′ or S4″), then $k.\star{\sim}i$.*

Definition 3.3. *(label function)*
Given $\mathcal{D}$ a formula or a sequence of formulas in $L_\Box$, and $\mathbf{I}$ a label set, we call $l\colon m(\mathcal{D}) \to \mathbf{I}$ a label function *on $\mathcal{D}$. Any nonempty set can be a label set.*

[3]We equate axiom schemes with their simplest propositional letter substitutional instances.

Each label function l on $\mathcal{D}$ induces an equivalence relation $\overset{l}{\sim}$ on $\mathcal{O}(\mathcal{D})$ such that for any $x, y \in \mathcal{O}(\mathcal{D})$, $x \overset{l}{\sim} y$ iff $\mathcal{D}(x) = \mathcal{D}(y)$, and for any $x.z$, $y.z \in m(\mathcal{D})$, $l(x.z) = l(y.z)$.

Definition 3.4 (proof label function)**.**

A label function l on a proof $\mathcal{D}$ in S4, S4′ or S4″ is a proof label function *if the induced equivalence relation $\overset{l}{\sim}$ is a proof equivalence relation.*

Definition 3.5. *For two label functions l, l' on $\mathcal{D}$, we say l' covers l if for any x, y in $\mathcal{O}(\mathcal{D})$, $x \overset{l'}{\sim} y$, whenever $x \overset{l}{\sim} y$.*

Lemma 3.6. *If l is a proof label function on a proof $\mathcal{D}$, and l' is a label function on $\mathcal{D}$ such that l' covers l, then l' is also a proof label function on $\mathcal{D}$.*

Lemma 3.7. *Given a proof $\mathcal{D}$, and label functions l, l' on $\mathcal{D}$, if for any $x, y \in m(\mathcal{D})$ with $x \overset{l}{\sim} y$, $l'(x) = l'(y)$, then l' covers l.*

According to the definition, a proof can be supplied with more than one proof label function. A proof label function can be as coarse as the label function which assigns the same label to every modal occurrence. But the finer the proof label function with respect to the relation of covering, the more essence of the structure of the proof preserved in the proof label function.

As our earlier observation of the unrealizable proof fragment shows, what matters is some specific relations among m-formula occurrences in modal axioms. We will call the collection of the relations in concern the *stamp* of the modal logical system.

Definition 3.8 (the standard stamp of S4)**.**

A stamp $\mathcal{A}$ of a modal logical system is a collection of binary relations $\xrightarrow{A}$ on $m(A)$ with A a modal axiom scheme.

The standard stamp of S4 and S4′ include (the scheme names stand for the schemes):

$\xrightarrow{A1} = \{ \langle a, b.b \rangle, \langle b.a, b.b \rangle \}$,

$\xrightarrow{A2} = \{ \langle a, b \rangle \}$,

and, one more for S4″:

$\xrightarrow{A4} = \{ \langle a, b \rangle \}$.

That is, the standard stamp is concerned with the directed relations from $\Box(F \to G)$ to $\Box G$ and from $\Box F$ to $\Box G$ in the axiom scheme $\Box(F \to G) \to$

$(\Box F \rightarrow \Box G)$, from $\Box F$ to $\Box(\Box F)$ in $\Box F \rightarrow \Box(\Box F)$, and from the first $\Box F$ to the second $\Box F$ in $\Box F \rightarrow \Box F$.

Below $[x]^l$ denotes the equivalence class induced by the label function l and containing the occurrence x.

Definition 3.9. *Given a proof label function l on $\mathcal{D}$ in a system with stamp $\mathcal{A}$, $\xrightarrow{\mathcal{A}}$ is a relation defined on the equivalence classes induced by l such that if $\mathcal{D}(k)$ is an axiom instance of an axiom scheme A and $x \xrightarrow{A} y$ with $\xrightarrow{A} \in \mathcal{A}$, then $[k.x]^l \xrightarrow{\mathcal{A}} [k.y]^l$.*

A chain of l-induced equivalence classes, $E_1, E_2, \ldots, E_n$, with respect to the relation $\xrightarrow{\mathcal{A}}$, namely, $E_i \xrightarrow{\mathcal{A}} E_{i+1}$ for any E_i in the chain, is circular, if there are $i \neq j$, $E_i = E_j$.

Now we give the formal definition of non-circular proof.

Definition 3.10 (non-circular proof)**.**

A proof label function l on a proof $\mathcal{D}$ in a system with stamp $\mathcal{A}$ is $\mathcal{A}$-circular if there is a chain of l-induced equivalence classes with respect to $\xrightarrow{\mathcal{A}}$ which is circular; otherwise l is $\mathcal{A}$-non-circular.

A proof $\mathcal{D}$ is $\mathcal{A}$-non-circular if we can define an $\mathcal{A}$-non-circular proof label function on $\mathcal{D}$.

So given a modal logical system, such as S4, there is more than one stamp that can be defined. The standard stamp that we are going to discuss is not the only one. But what is interesting about this stamp, and others for S4′ and S4″, is that we can be sure that the classes of non-circular proofs with respect to these stamps are complete. A proof of this result is the aim of the next section.

We will skip mentioning the stamps or what are the stamps $\mathcal{A}$ referring to in the ensuing discussions, which are presumed to be the standard as defined in Definition 3.8.

4 S4$^{\Delta}$ and the Completeness of Non-Circular Proofs

4.1 S4 and S4$^{\Delta}$

The main goal of this section is to establish the completeness of non-circular proofs, and for this purpose, we introduce logical systems S4$^{\Delta}$, and its variants. There are several reasons for this introduction. First of all, in the above section we give a general definition of non-circular proof with the label function as an auxiliary tool. The non-circularity of proofs essentially depends on the relations between m-formulas in the proof. However, it is possible to employ a structural label set to detect the non-circularity of proofs, and the set of natural number is a good candidate. Secondly, it is always easier to work on formulas with their labels built in as part of the formulas, instead of to work on formula occurrences and then consider their labels. Finally, since $S4^{\mathit{\Delta}}$ is introduced as a logical system, hence logical techniques can be applied to deal with the relevant problems. We need more properties of proof label function.

Definition 4.1 (increasing proof label function)**.**
A numerical proof label function $\mathit{\Delta}\colon m(\mathcal{D}) \to \mathbb{N}$ *on a proof* $\mathcal{D}$ *in S4, S4′, or S4″ is increasing if* $\mathit{\Delta}(k.x) < \mathit{\Delta}(k.y)$*, for any substitutional instance* $\mathcal{D}(k)$ *of an axiom scheme A, and any occurrences* x, y *with* $x \xrightarrow{A} y$*, where* $\xrightarrow{A} \in \mathcal{A}$*.*

Lemma 4.2. $\mathit{\Delta}$ *is an increasing proof label function on a proof* $\mathcal{D}$*, if and only if for any* $x,\ y \in \mathcal{O}(\mathcal{D})$*, if* $[x]^{\mathit{\Delta}} \xrightarrow{\mathcal{A}} [y]^{\mathit{\Delta}}$*, then* $\mathit{\Delta}(x) < \mathit{\Delta}(y)$*.*

Proof. By the definitions of increasing proof label function and the fact that if $w \overset{\mathit{\Delta}}{\sim} z$, then $\mathit{\Delta}(w) = \mathit{\Delta}(z)$. ⊣

Proposition 4.3. *A proof* $\mathcal{D}$ *is non-circular if and only if there exists an increasing proof label function* $\mathit{\Delta}$ *on* $\mathcal{D}$*.*

Proof. When $\mathit{\Delta}$ is increasing, by the Lemma 4.2, it immediately follows that every chain of $\mathit{\Delta}$-induced equivalence classes with respect to $\xrightarrow{\mathcal{A}}$ is non-circular. Hence every increasing proof label function is non-circular. On the other hand, when $\mathcal{D}$ is non-circular, there exists a proof label function l on $\mathcal{D}$ such that every chain of l-induced equivalence classes with respect to $\xrightarrow{\mathcal{A}}$ is non-circular. Let S be the set of these chains. S will be a finite set of finite chains. We define the function $\mathit{\Delta}_l\colon m(\mathcal{D}) \to \mathbb{N}$ such that $\mathit{\Delta}_l(x) = \max\{i \mid [x]^l \text{ is the } i\text{-th element of a chain in } S\}$. Since $\mathit{\Delta}_l(x) = \mathit{\Delta}_l(y)$ for any $x \overset{l}{\sim} y$ in $m(\mathcal{D})$, $\mathit{\Delta}_l$ covers l and hence $\mathit{\Delta}_l$ is a proof label function. Also, since $\mathit{\Delta}_l(x) < \mathit{\Delta}_l(y)$ for any $[x]^l \xrightarrow{\mathcal{A}} [y]^l$, $\mathit{\Delta}_l$ is increasing. This completes the proof. ⊣

Given a non-circular proof label function l on a proof $\mathcal{D}$, we can actually build an increasing proof label function such that different *initial equivalence classes*, equivalence classes without predecessors, have different number labels.

Definition 4.4. *An equivalence class* $[x]^l$ *is* initial *if and only if there's no* $[y]^l$ *such that* $[y]^l \xrightarrow{\mathcal{A}} [x]^l$.

Corollary 4.5. *Let* S *be the set of initial equivalence classes in* $\mathcal{D}$ *and* $f\colon S{\to}\mathbb{N}$. *There exists an increasing proof label function* Δ *covering* l *such that for any* $[x]^l{\in}S$, $\Delta(x){=}f([x]^l)$.

Proof. Let $d{=}\max\{f([x]^l) \mid [x]^l{\in}S\}$, and Δ_l be the increasing function built based on the procedure given in the above Proposition 4.3. Then the numerical function Δ such that for any $x{\in}m(\mathcal{D})$, $\Delta(x){=}f([x]^l)$ if $[x]^l$ is initial, otherwise $\Delta(x){=}\Delta_l(x){+}d$ will do the job. ⊣

Now we introduce a new family of languages. Let $\mathbf{I}$ be a label set. The language $L_{\mathbf{I}}$ is an extended propositional language with the following non-propositional formula formation rule: if $\phi \in L_{\mathbf{I}}$ and $u \in \mathbf{I}$, $\Box\phi^u \in L_{\mathbf{I}}$. We will also call a formula of the form $\Box\phi^u$ m-formula, and call the label u the principal label of the formula, denoted as $\nu(\Box\phi^u)$.

The Definition 3.1 can be well-adapted to define formula occurrences of formulas or sequences of formulas in $L_{\mathbf{I}}$. The only needed change is to take care of the new m-formulas: if $x \in \mathcal{O}(\phi)$ and $\phi(x){=}(\Box\psi^u)$, then $x.\star \in\mathcal{O}(\phi)$ and $\phi(x.\star){=}\psi$.

Later in this paper we will see several translations between formulas and proofs in $L_\Box$ and $L_{\mathbf{I}}$. If not stated otherwise, they are all presumed to *fix propositional letters* and *commute with boolean connectives*. In other words, the purpose of these translations is to add or remove labels, or to switch the labels from one to another. In fact, we can view a realization as a translation of this kind with proof terms as labels.

Let $\mathcal{D}$ be a sequence of formulas in $L_\Box$ and $\mathcal{F}$ be a sequence of formulas in $L_{\mathbf{I}}$ with $\mathbf{I}$ a label set.

Definition 4.6.

For any label function $l\colon m(\mathcal{D}) \to \mathbf{I}$, *an induced translation, also denoted as* l, *from* $\mathcal{D}$ *to a sequence of formulas* $\mathcal{D}^l$ *in* $L_{\mathbf{I}}$ *is such that for any* $x \in m(\mathcal{D})$, $\mathcal{D}^l(x) = \Box\mathcal{D}^l(x.\star)^u$ *with* $u = l(x)$.

A $\Box$*-translation on* $\mathcal{F}$ *is a translation such that for every* $x \in m(\mathcal{F})$, $\mathcal{F}^\Box(x) = \Box\mathcal{F}^\Box(x.\star)$,

$l_\mathcal{F}$ *is a label function on* $\mathcal{F}^\Box$ *induced by* $\mathcal{F}$ *such that for every* $x \in m(\mathcal{F}^\Box)$, $l_\mathcal{F}(x) = \nu(\mathcal{F}(x))$.

$l_\mathcal{F}$ is well-defined since $x{\in}m(\mathcal{F}^\Box)$ if and only if $x{\in}m(\mathcal{F})$, and we have the following equivalence results:

Lemma 4.7. *For x, $y \in \mathcal{O}(\mathcal{D})$, $x \overset{l}{\sim} y$ iff $\mathcal{D}^l(x) = \mathcal{D}^l(y)$.*

Proposition 4.8. *$\mathcal{F} = \mathcal{D}^l$ iff $\mathcal{F}^\Box = \mathcal{D}$ and $l_\mathcal{F} = l$.*

S4$^\Delta$ is a logical system defined on the set of formulas L_Δ, a case of $L_\mathbf{I}$ languages with natural numbers as labels. The system is the following:

Axiom Schemes:

A0 axiom schemes of classical propositional logic
A1 $\Box(F \to G)^i \to (\Box F^j \to \Box G^k)$, i, $j < k$
A2 $\Box F^i \to \Box(\Box F^i)^j$, $i < j$
A3 $\Box F^i \to F$

Inference rules

R1 $F, F \to G \vdash G$ "modus ponens"
R2 $\vdash \Box F^i$ for any i, if $\vdash F$ "necessitation"

System S4$'^\Delta$ is S4$^\Delta$ with *necessitation* replaced by *axiom necessitation* "$\vdash \Box F^i$ for any i, if $\vdash F$ and F is an axiom," and S4$''^\Delta$ is S4$'^\Delta$ with the addition of the axiom scheme A4, "$\Box F^i \to \Box F^j, i<j$."

An interesting and apparent feature of the system is that for a formula being an axiom instance, the number labels in the formula have to satisfy some condition, and this feature is just the key to the the success of the following theorem concerning the formal relations between proofs in variant S4-systems and S4$^\Delta$-systems.

Theorem 4.9. *A proof $\mathcal{D}$ in S4, S4$'$, or S4$''$ is non-circular if and only if there is a proof label function $\Delta\colon m(\mathcal{D}) \to \mathbb{N}$ such that $\mathcal{D}^\Delta$ is a proof in S4$^\Delta$, S4$'^\Delta$, or S4$''^\Delta$, respectively.*

Proof. Given $\mathcal{D}^\Delta = \mathcal{F}$ a proof in S4$^\Delta$ [S4$'^\Delta$, S4$''^\Delta$], it is not difficult to check that $\mathcal{D}$ ($=\mathcal{F}^\Box$) is a proof in S4 [S4$'$, S4$''$] and that Δ ($=l_\mathcal{F}$) is an increasing proof label function on $\mathcal{D}$ since it needs to satisfy the numerical conditions set on the modal axiom schemes of the system. Hence $\mathcal{D}$ is non-circular. For the other direction, suppose that $\mathcal{D}$ is a non-circular proof in S4$^\Delta$ [S4$'^\Delta$, S4$''^\Delta$]. By Proposition 4.3, an increasing proof label function Δ defined on $\mathcal{D}$ exists, and hence all we need to do is to check if $\mathcal{D}^\Delta$ is a proof in S4$^\Delta$ [S4$'^\Delta$, S4$''^\Delta$]. Since Δ is a proof label function, then it can be sure by Lemma 4.7 that whenever ϕ is an axiom or derived by a rule application in $\mathcal{D}$, so is ϕ^Δ in $\mathcal{D}^\Delta$ except that ϕ is a modal axiom. But since Δ is also increasing, conditions on modal axioms will be fulfilled, and hence $\mathcal{D}^\Delta$ is a proof in S4$^\Delta$ [S4$'^\Delta$, S4$''^\Delta$]. ⊣

Corollary 4.10. *$\mathcal{F} = \mathcal{D}^\Delta$ is a proof in S4$^\Delta$, S4$'^\Delta$, or S4$''^\Delta$, if and only if Δ ($=l_\mathcal{F}$) is an increasing proof label function on $\mathcal{D}$ ($=\mathcal{F}^\Box$).*

4.2 Completeness of Non-Circular Proofs

As mentioned in the introduction, the long-term goal of this project is to establish a direct procedure turning Hilbert-style circular proofs into non-circular ones, which the completeness of non-circular proofs immediately follows. Right now we will deal with the problem by analyzing Gentzen-style proofs. We first supply the Gentzen systems that corresponds to S4 and $S4^{\Delta}$ respectively. Here are some notations. A sequent $\Gamma \Rightarrow \Gamma'$ is a pair of finite multisets Γ, Γ' of formulas. It is convenient for us to view a sequent as a formula $C_1 \to (\ldots \to (C_n \to \bigvee \Gamma')\ldots)$. Given a multiset $\Gamma = \{C_i\}$ of formulas in $L_{\Box}$, $\Box\Gamma = \{\Box C_i\}$. Given a multiset $\Gamma = \{C_i\}$ of formulas in L_{Δ}, $\Box\Gamma^{\iota} = \{\Box C_i^{j_i}\}$, for j_i a number in the multiset ι. $|\Gamma|$ is the number of formulas in Γ. The Gentzen system S4G is:

The only axiom is that $P \Rightarrow P$, for a propositional letter P.

The rules for weakening (W) and contraction (C)

$$\text{LW}\ \frac{\Gamma \Rightarrow \Gamma'}{A, \Gamma \Rightarrow \Gamma'}\ ,\quad \text{RW}\ \frac{\Gamma \Rightarrow \Gamma'}{\Gamma \Rightarrow \Gamma', A}$$

$$\text{LC}\ \frac{A, A, \Gamma \Rightarrow \Gamma'}{A, \Gamma \Rightarrow \Gamma'}\ ,\quad \text{RC}\ \frac{\Gamma \Rightarrow \Gamma', A, A}{\Gamma \Rightarrow \Gamma', A}$$

The classical logical rules (i=0,1):

$$\text{L}\neg\ \frac{\Gamma \Rightarrow \Gamma', A}{\neg A, \Gamma \Rightarrow \Gamma'}\ ,\quad \text{R}\neg\ \frac{\Gamma, A \Rightarrow \Gamma'}{\Gamma \Rightarrow \Gamma', \neg A}$$

$$\text{L}\wedge\ \frac{A_i, \Gamma \Rightarrow \Gamma'}{A_0 \wedge A_1, \Gamma \Rightarrow \Gamma'}\ ,\quad \text{R}\wedge\ \frac{\Gamma \Rightarrow \Gamma', A \quad \Gamma \Rightarrow \Gamma', B}{\Gamma \Rightarrow \Gamma', A \wedge B}$$

$$\text{L}\vee\ \frac{A, \Gamma \Rightarrow \Gamma' \quad B, \Gamma \Rightarrow \Gamma'}{A \vee B, \Gamma \Rightarrow \Gamma'}\ ,\quad \text{R}\vee\ \frac{\Gamma \Rightarrow \Gamma', A_i}{\Gamma \Rightarrow \Gamma', A_0 \vee A_1}$$

$$\text{L}\to\ \frac{\Gamma \Rightarrow \Gamma', A \quad B, \Gamma \Rightarrow \Gamma'}{A \to B, \Gamma \Rightarrow \Gamma'}\ ,\quad \text{R}\to\ \frac{A, \Gamma \Rightarrow \Gamma', B}{\Gamma \Rightarrow \Gamma', A \to B}$$

The modal rules:

$$\text{L}\Box\ \frac{A, \Gamma \Rightarrow \Gamma'}{\Box A, \Gamma \Rightarrow \Gamma'}\ ,\quad \text{R}\Box\ \frac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ .$$

S4G as listed above is similar to the propositional fragment of **G1s** in [17], except that it is a system for the language with single modality $\Box$, and with the negation $\neg$ instead of the falsehood $\bot$. It is therefore complete with respect to the standard Hilbert-style system of S4.

$S4^{\Delta}G$ is a Gentzen-style proof system defined on formulas in L_{Δ}. Its axiom and rules are the same as the ones in S4G except that its modal rules are the following:

$$\text{L}\Box\ \frac{A, \Gamma \Rightarrow \Gamma'}{\Box A^i, \Gamma \Rightarrow \Gamma'}\ ,\quad \text{for any } i$$

R$\Box$ $\dfrac{\Box\Gamma^{\iota} \Rightarrow A}{\Box\Gamma^{\iota} \Rightarrow \Box A^{i}}$, for any $i > \max(\iota) + |\Gamma|$, when $|\Gamma| \neq 0$, and
for any i when $|\Gamma| = 0$.

There are two main steps in our procedure of proving the completeness of non-circular proofs. The first step is to show that every S4G proof can be turned into an S4$^{\Delta}$G proof, and the second is that S4$^{\Delta}$G is sound with respect to S4$^{\Delta}$. In the following, when we adjust m-formula occurrences' number labels in an S4$^{\Delta}$G proof, we adjust all the related formulas of premises and conclusions of rules to the same number. Obviously S4G is a cut-free system, and recall that cut-free proofs respect the polarity of formulas.

Lemma 4.11. *If in an S4$^{\Delta}$G proof we adjust the number labels such that the principal labels of negative m-formula occurrences become smaller, and those of positive m-formula occurrences become larger, the result will still be an S4$^{\Delta}$G proof.*

Proof. The only applications of inference rules will be affected are the applications of the right modal rule. However, the numerical condition on the rule is still fulfilled after the adjustment. ⊣

Proposition 4.12. *Every S4G proof $\mathcal{G}$ can be translated to a proof $\mathcal{G}^{\Delta}$ in S4$^{\Delta}$G by providing suitable numerical labels for m-formula occurrences in $\mathcal{G}$.*

Proof. The proof is quite straightforward. We can give suitable labels to an S4G proof by induction on the depth of the prooftree. There are some cases, like applications of two-premise inference rules, in which the labels need adjustments. In these cases, we can apply the previous lemma to adjust the number labels in the premises of an application and the prooftrees above the premises such that the labels of m-formulas in the premises which relate to the same formula in the conclusion match to each other. Since S4G is cut-free, so this always can be done, and then the two-premise inference rules of S4$^{\Delta}$G can be well applied.

Nevertheless, there exists a very efficient method. We can just let all negative formula occurrences have the label 0, and all positive formula occurrences have the label equal to the number of m-formula occurrences in the S4G proof. Then the numerical conditions on all the applications of the right modal rules will be satisfied. ⊣

Proposition 4.13. *Every S4$^{\Delta}$G proof can be converted to an S4$^{\Delta}$ proof with the same conclusion.*

Proof. The procedure is to convert each application of an inference rule (including axioms) to a sequence of formulas. For the propositional part, we can pick up the procedure listed in [6] and for the applications of the left modal rule L$\Box$, the translation is not difficult to figure out. Here we only check that there is such a conversion for applications of the right modal rules, R$\Box$. We need the following lemma:

Lemma 4.14. *For* $|\Gamma| > 0$ *and* $i > \max(\max(\iota) + 1, e) + |\Gamma| - 1$, $\Box(\Box\Gamma^\iota \Rightarrow A)^e \to (\Box\Gamma^\iota \Rightarrow \Box A^i)$ *is provable in S4*$^\Delta$*.*

Proof. It's equivalent to prove that for any $|\Theta| \geq 0$ and $i > \max(\max(\iota, j) + 1, e) + |\Theta|$,

$(*)$ $\Box(\Box C^j \to (\Box\Theta^\iota \Rightarrow\ A))^e \to (\Box C^j \to (\Box\Theta^\iota \Rightarrow \Box A^i))$

is provable in S4$^\Delta$. We will prove this by induction on $|\Theta|$. Noticed that for any multiset Θ, if number $e' > max(e, j + 1)$, then

$\Box(\Box C^j \to (\Box\Theta^\iota\Rightarrow A))^e \to (\Box(\Box C^j)^{j+1} \to \Box(\Box\Theta^\iota\Rightarrow A)^{e'})$

is an A1 axiom, and

$\Box C^j \to \Box(\Box C^j)^{j+1}$ is an A2 axiom, and therefore

$(**)$ $\Box(\Box C^j \to (\Box\Theta^\iota \Rightarrow\ A))^e \to (\Box C^j \to \Box(\Box\Theta^\iota \Rightarrow A)^{e'})$

is provable in S4$^\Delta$.

When Θ is empty, let $e' = i > \max(e, j+1)$. Then $(**)$ holds, and hence the base case of $(*)$ is proved. For the induction step, suppose $|\Theta| = k+1$ and $\Box\Theta^\iota = \Box\Theta'^{\iota'} \cup \Box C'^{j'}$. Let $e' = \max(j+1, e) + 1$, and hence $(**)$ holds. Now since $i > \max(\max(\iota, j)+1, e)+|\Theta| \geq \max(\max(\iota', j')+1, e')+|\Theta'|$, by Induction Hypothesis, $\Box(\Box\Theta^\iota \Rightarrow\ A)^{e'} \to (\Box\Theta^\iota \Rightarrow \Box A^i)$, which is equivalent to $\Box(\Box C'^{j'} \to (\Box\Theta'^{\iota'} \Rightarrow\ A))^{e'} \to (\Box C'^{j'} \to (\Box\Theta'^{\iota'} \Rightarrow \Box A^i))$, holds. Then by classical propositional logic, $(*)$ is provable in S4$^\Delta$. This finishes the induction step and the proof. ⊣

Since if $\Box\Gamma^\iota\Rightarrow A$ is provable in S4$^\Delta$, when Γ is empty, by *necessitation*, $\Rightarrow\Box A^i$ is provable for any i, and when Γ is not empty, $\Box(\Box\Gamma^\iota\Rightarrow A)^0$ is provable, and then, following the procedure in the previous lemma, we can produce an S4$^\Delta$ proof for $\Box\Gamma^\iota\Rightarrow\Box A^i$, whenever $i > \max(\iota)+|\Gamma|$. This completes the proof of Proposition 4.13. ⊣

Now we can prove one of few structural properties concerning Hilbert-style proofs.

Theorem 4.15. *Every S4 theorem has a non-circular proof.*

Proof. Let ϕ be an S4 theorem. Since S4G is complete, an S4G proof $\mathcal{G}$ of ϕ exists. Then following Proposition 4.12, we can turn the S4G proof into an S4$^{\Delta}$G proof $\mathcal{G}^{\Delta}$ by assigning suitable numerical labels to m-subformulas. Now following the procedure in Proposition 4.13, we can translate the S4$^{\Delta}$G proof into an S4$^{\Delta}$ proof $\mathcal{F}$. Then $\mathcal{F}^{\Box}$ is a non-circular proof of ϕ. Moreover, $\mathcal{F}$ is the Hilbert-style proof translated from $\mathcal{G}$ by the procedure similar to the one in Proposition 4.13 but without the need of concerning numerical labels. $\dashv$

We also have the realization theorem for S4$^{\Delta}$:

Corollary 4.16. *For a formula $\phi \in L_{\Box}$, ϕ is an S4 theorem if and only if there is a numerical label function Δ on ϕ such that ϕ^{Δ} is an S4$^{\Delta}$ theorem.*

4.3 From S4$^{\Delta}$ to S4$'^{\Delta}$

The aim of this subsection is to provide an algorithm that translates S4$^{\Delta}$ proofs into S4$'^{\Delta}$ proofs. This algorithm is needed as an intermediate step for the realization theorem for LP, and also helps to establish the Δ-version of realization theorem for S4$'^{\Delta}$ and S4$''^{\Delta}$. We will provide two methods: one is called inductive and the other structural. The structural is efficient but limited to generalize.

We need to do some preliminary work. First, we will presuppose that in the S4$^{\Delta}$ proof in discussion every *R2-formula*, the formula derived by necessitation, is initial. This is the case when the proof is translated from an S4$^{\Delta}$G proof by the procedure given above. However, in general if an R2-formula $\Box\phi^i$ has predecessors, we can extend the proof by adding formulas including

$\Box\phi^0$, $\phi\to\phi$, $\Box(\phi\to\phi)^0$, $\Box(\phi\to\phi)^0\to(\Box\phi^0\to\Box\phi^i)$, $\Box\phi^0\to\Box\phi^i$,

and a proof of the tautology $\phi \to \phi$ if it is not an axiom.

Second, since now in our proof every R2-formula is initial, we can adjust the labels in the proof such that the number labels of these initial formulas have the numbers we would like them to have, as suggested by Corollary 4.5. In the following, given an S4$^{\Delta}$ proof, before we extend the proof to an S4$'^{\Delta}$ proof, we will firstly modify the number labels such that if $\Box\phi^i$ is an R2-formulas derived from the k-th element of the proof, then $i{=}k$ for the structural method, and $i{=}4^jk$ for the inductive method, where $\Box\phi^i$ is the conclusion of the j-th application of the necessitation rule.

We first see the inductive method. A lemma is in order. Here are some notations. Let $\mathcal{F}$ be an S4$^{\Delta}$ or S4$'^{\Delta}$ proof, and $l\colon \mathcal{F} \to \mathbb{N}$ be the length function of $\mathcal{F}$ ($l(\phi) = k$ provided ϕ is the k-th element of $\mathcal{F}$). We call $g\colon \mathcal{F} \to \mathbb{N}$ a *super-length function* on $\mathcal{F}$ if for every $\phi, \psi \in \mathcal{F}$, $0 \leq g(\phi) - l(\phi)$ and $g(\phi) - l(\phi) \leq g(\psi) - l(\psi)$ provided $l(\phi) \leq l(\psi)$. We call $\mathcal{F}$ *regular with respect to* g if $j \leq 4g(A)$ for any formula $\Box A^j$ dervied from A by axiom necessitation.

Lemma 4.17. *If $\mathcal{F}$ is an $S4'^{\Delta}$ proof regular with respect to a super-length function g, then there exists an $S4'^{\Delta}$ proof $\mathcal{F}'$ such that for every formula ϕ in $\mathcal{F}$, $\Box\phi^{4g(\phi)}$ is in $\mathcal{F}'$. Furthermore, $\mathcal{F}'$ is regular with respect a super-length function g', where for any formula $\phi \in \mathcal{F}$, $g'(\phi) = 4g(\phi)$, and if ψ is the conclusion of $\mathcal{F}$, $g'(\Box\psi^{4g(\psi)}) = 4(g(\psi)+1)$.*

Proof. We will construct an $\mathrm{S4}'^{\Delta}$ proof $\mathcal{F}'$ by inductively adding up to three formulas after each formula ϕ of $\mathcal{F}$ such that $\Box\phi^{4g(\phi)}$ is added. If ϕ is an axiom, then add $\Box\phi^{4g(\phi)}$. If ϕ is derived from $\psi{\to}\phi$ and ψ, then add formulas $\Box(\psi{\to}\phi)^j{\to}(\Box\psi^k{\to}\Box\phi^i)$, $\Box\psi^k{\to}\Box\phi^i$, $\Box\phi^i$ after ϕ with $j{=}4g(\psi{\to}\phi)$, $k{=}4g(\psi)$, and $i{=}4g(\phi)$. Finally, if $\phi \equiv \Box A^j$ is derived from A by axiom necessitation, add formulas $\Box A^j{\to}\Box(\Box A^j)^i$, $\Box(\Box A^j)^i$ with $i{=}4g(\phi)$, which is larger than j since $\mathcal{F}$ is regular. Then it can be checked that $\mathcal{F}'$ is an $\mathrm{S4}'^{\Delta}$ proof. Also, let $g'\colon \mathcal{F}' \to \mathbb{N}$ be the function such that $g'(\phi){=}4g(\phi)$ if $\phi{\in}\mathcal{F}$, $g'(\psi){=}4g(\phi)+i$ if ψ is not the conclusion of $\mathcal{F}$ and is the i-th formula to be added right after ϕ in the procedure ($i \leq 3$), and $g'(\Box\psi^{4g(\psi)}) = 4(g(\psi)+1)$ if ψ is the conclusion of $\mathcal{F}$. Then $\mathcal{F}'$ is is regular with respect to g'. ⊣

Proposition 4.18. *(inductive method) Every $S4^{\Delta}$ proof $\mathcal{F}$ can be extended to be a proof in $S4'^{\Delta}$.*

Proof. Let l be the length function of $\mathcal{F}$, and $\mathcal{F}_\phi$ denote the initial segment of $\mathcal{F}$ up to ϕ. It is sufficient to prove that for every formula $\Box\phi^{4^j k}$ derived from the k-th element ϕ by the j-th application of necessitation, $\mathcal{F}_{\Box\phi^{4^j k}}$ can be extended to an $\mathrm{S4}'^{\Delta}$ proof $\mathcal{F}_j$ of $\Box\phi^{4^j k}$ with $\mathcal{F}_j$ regular with respect to a super-length function g_j such that for any formula $\psi{\in}\mathcal{F}_{\Box\phi^{4^j k}}$, $g_j(\psi){=}4^j l(\psi)$. The proof is by induction on j, and to simplify the discussion, $l(\Box\phi^{4^j k})$ is assumed to be $l(\phi){+}1$. Suppose $\Box\phi^{4k}$ is derived from ϕ by the first application of necessitation, then $\mathcal{F}_\phi$ is also an $\mathrm{S4}'^{\Delta}$ proof of ϕ and regular with respect to its length function, and hence, by Lemma 4.17, $\mathcal{F}_\phi$ can be extended to an $\mathrm{S4}'^{\Delta}$ proof $\mathcal{F}_1$ of $\Box\phi^{4k}$ regular with respect to a function g_1 such that $g_1(\psi) = 4l(\psi)$ for any $\psi{\in}\mathcal{F}_{\Box\phi^{4k}}$. The base case is proved. Now suppose $j{>}1$ and $\Box\psi^{4^{j-1}s}$ is derived from ψ by the $(j{-}1)$-th application of necessitation. By Induction Hypothesis, there is an $\mathrm{S4}'^{\Delta}$ proof $\mathcal{F}_{j-1}$

of $\Box\psi^{4^{j-1}s}$ regular with respect to a super-length function g_{j-1} such that $g_{j-1}(\theta){=}4^{j-1}l(\theta)$ for any $\theta{\in}\mathcal{F}_{\Box\psi^{4^{j-1}k}}$. Now we can append in order formulas which are not in $\mathcal{F}_{\Box\psi^{4^{j-1}k}}$ but in $\mathcal{F}_\phi$ to $\mathcal{F}_{j-1}$ to form an $\mathrm{S4}'^{\Delta}$ proof of ϕ regular with respect to a function g' such that $g'(\theta){=}g_{j-1}(\theta)$ if $\theta{\in}\mathcal{F}_{j-1}$ and $g'(\theta){=}4^{j-1}l(\theta)$ otherwise. Then applying Lemma 4.17 again, the induction step and the proof is complete. ⊣

Proposition 4.19. *(structural method) Every $S4^{\Delta}$ proof $\mathcal{F}$ can be extended to a proof in $S4'^{\Delta}$.*

Proof. We will lengthen the proof $\mathcal{F}$ inductively such that either the formula $\Box\phi^i$ will be added after the i-th non-conclusion formula ϕ of the proof, or if $\Box\phi^i$ is an R2-formula and ϕ is not an axiom, we can redirect it such that it is still derivable in the lengthened proof but not from ϕ by necessitation rule anymore. Then the resulting sequence is an $\mathrm{S4}'^{\Delta}$ proof. If ϕ is an axiom, we add $\Box\phi^i$ right after ϕ. If ϕ is derived from the j-th element $\psi{\to}\phi$ and k-th element ψ, we add formulas $\Box(\psi{\to}\phi)^j{\to}(\Box\psi^k{\to}\Box\phi^i)$ and $\Box\psi^k{\to}\Box\phi^i$ after ϕ, and then either add $\Box\phi^i$, or not if $\Box\phi^i$ is an R2-formula in $\mathcal{F}$, such that it is derived from $\Box\psi^k{\to}\Box\phi^i$ and $\Box\psi^k$, which has been in the proof by induction. Finally, if $\phi \equiv \Box\psi^j$ is derived from the j-th formula ψ in $\mathcal{F}$, and now is derived from some other formulas if ψ is not an axiom, we add formulas $\Box\psi^j{\to}\Box(\Box\psi^j)^i$ after ϕ, and then add $\Box(\Box\psi^j)^i$, or not if it has been in $\mathcal{F}$ derived from $\Box\psi^j$ by necessitation, such that it is derived from $\Box\psi^j$ and $\Box\psi^j{\to}\Box(\Box\psi^j)^i$. This completes the proof. ⊣

The efficiency of the structural method over the inductive is suggested by the numerical labels employed in the proofs (check the proofs in the appendices), but the former method only works for systems with the A2 transitivity axiom. One of the reasons for the introduction of the two methods is to show that our overall procedure for the realization theorem for LP, which will be completed in the next section, can be generalized to concern the LP-counterparts of other modal logics as well. In the inductive method, only the A2 axiom instances of the form that $\Box A^i{\to}\Box(\Box A^i)^j$ with A an axiom are used, and these instances are still realizable in the LP-counterparts of modal logical systems without the transitivity axiom.[4]

That we carefully elaborate the numerical labels in the proofs is to show that it is possible to use Δ-like logics to study the lengths of proofs. The

[4]These LP-systems can be found in [15, 10], in which the rule of *strong axiom necessitation*: "R2*: $\vdash c{:}F$ for $c{\in}\mathcal{C}$, if $\vdash F$ and F is an axiom or is inferable using R2*", instead of the rule of necessitation, is employed.

original proof of the realization theorem for LP provided in [2] is based on a direct translation from cut-free Gentzen-style S4 proofs to LP proofs. The difficulty of such a procedure rests on the construction of suitable proof terms for modal occurrences through the analysis of the Gentzen-style proof tree, especially dealing with the applications of the right modal rule. It is proved in [5] that the original realization procedure will produce an LP proof with length exponential to the size of the initial cut-free Gentzen-style proof, and can be improved such that only proofs with polynomial length are generated. Among other technical details, the improvement, where the instances of the A2 axiom of LP, $s{:}\phi \to !s{:}(s{:}\phi)$, play an important role, however, can be analyzed as with the same idea as the one in the improvement made by adopting the structure method over the inductive method in a procedure of generating $\mathrm{S4}'^{\Delta}$ proofs from $\mathrm{S4}^{\Delta}$ proofs, or in a similar but simplified procedure of generating S4′ proofs from S4 proofs.

The discussions in this subsection also imply that every non-circular proof of S4 can be extended to a non-circular proof of S4′, or S4″, a super-system of S4′, and hence imply that the realization theorem also holds for $\mathrm{S4}'^{\Delta}$ and $\mathrm{S4}''^{\Delta}$.

Corollary 4.20. *Every S4′ (S4″) theorem has a non-circular S4′ (S4″) proof, and for a formula $\phi \in L_{\Box}$, ϕ is an S4′ (S4″) theorem if and only if there is a numerical label function Δ such that ϕ^{Δ} is an $S4'^{\Delta}$ ($S4''^{\Delta}$) theorem.*

5 Proof Realization and the Realization Theorem for LP

5.1 Proof Realization

Now we discuss the *proof realization* between proofs in the variant S4-systems and their explicit LP-counterparts, and what is left for us is to design a procedure that can translate between proofs in $\mathrm{S4}^{\Delta}$-systems and proofs in their corresponding LP-systems. As we can see, with the guide of the number labels, a quite efficient procedure can be established.

Definition 5.1 (proof assignment)**.**

Given a sequence of formulas $\mathcal{F}$ in L_{Δ}, a proof assignment p on $\mathcal{F}$ assigns each pair (ϕ, i) for a subformula $\Box\phi^i$ in $\mathcal{F}$ a proof term.

Each proof assignment p induces a translation from $\mathcal{F}$ to a sequence of formulas in $L_{:}$ such that $(\Box\phi^i)^p = p(\phi, i){:}(\phi^p)$.

Here's a classification of m-formulas in a proof.

Definition 5.2. *Given a proof $\mathcal{F}$ in one of the $S4^{\Delta}$ systems,*

1. *if axiom $\Box(\phi\to\psi)^i\to(\Box\phi^j\to\Box\psi^k)$ is in $\mathcal{F}$, we call $\Box\psi^k$ A1-formula and the leading formula of the axiom, with $\Box(\phi\to\psi)^i$, $\Box\phi^j$, and ordered pair of formulas $\langle\Box(\phi\to\psi)^i, \Box\phi^j\rangle$ as its α-predecessor, β-predecessor, and A1-predecessor pair, respectively,*
2. *if axiom $\Box\phi^i\to\Box(\Box\phi^i)^j$ is in $\mathcal{F}$, we call $\Box(\Box\phi^i)^j$ A2-formula and the leading formula of the axiom, with $\Box\phi^j$ as its γ-predecessor,*
3. *if axiom $\Box\phi^i\to\Box\phi^j$ is in $\mathcal{F}$, we call $\Box\phi^j$ A4-formula and the leading formula of the axiom, with $\Box\phi^i$ as its δ-predecessor,*
4. *if $\Box\phi^i$ in $\mathcal{F}$ is derived by* necessitation *(in the case of $S4^{\Delta}$), or by* axiom necessitation *(in the case of $S4'^{\Delta}$ or $S4''^{\Delta}$), we call $\Box\phi^i$ R2-formula.*

Definition 5.3. *An m-formula in a proof can fall into more than one of the categories given in the previous definition. If a formula is in at most one of the above categories we say the formula is* stable. *Especially, we call an m-formula, say,* A1-stable, *if it is an A1-formula only, or does not belongs to any of the above categories.*

In the following definition, the notations of $o(s)$ and $\dot{o}(s)$ for some proof term s are what we used in the introduction of the systems ELP.

Definition 5.4 (characteristic proof assignment)**.**
Each proof $\mathcal{F}$ in $S4^{\Delta}$, $S4'^{\Delta}$ or $S4''^{\Delta}$ will be associated with a system of equations $\mathbf{E}_{\mathcal{F}}$ for an unknown proof assignment p. The system consists of:

1. *$p(\phi, i){=}o(p(\psi, j)\cdot p(\theta, k))$ when $\langle\Box\psi^j, \Box\theta^k\rangle$ is a predecessor pair of $\Box\phi^i$,*
2. *$p(\phi, i){=}o(!p(\psi, j))$ when $\Box\psi^j$ is a γ-predecessor of $\Box\phi^i$,*
3. *$p(\phi, i){=}\dot{o}(p(\psi, j))$ when $\Box\psi^j$ is a δ-predecessor of $\Box\phi^i$.*
4. *$p(\phi, i){=}o(c)$, for some $c \in \mathcal{C}$, when $\Box\phi^i$ is an R2-formula. We will call these the R2-equations of $\mathbf{E}_{\mathcal{F}}$.*

A characteristic proof assignment *of $\mathcal{F}$ is a proof assignment which satisfies all the equations in $\mathbf{E}_{\mathcal{F}}$. And we call a characteristtic proof assignment* simple, *if it satisfies the minimal requirement of each condition, that is, for example, if $\langle\Box\psi^j, \Box\theta^k\rangle$ is a predecessor pair of $\Box\phi^i$, $p(\phi, i){=}p(\psi, j)\cdot p(\theta, k)$, and if $\Box\psi^j$ is a δ-predecessor of $\Box\phi^i$, $p(\phi, i){=}p(\psi, j){+}t$ or $p(\phi, i){=}t{+}p(\psi, j)$ for some proof term t.*

The definitions of stable formula and simple characteristic proof assignment are needed for the next subsection.

Lemma 5.5. *Each proof $\mathcal{F}$ in $S4^{\Delta}$, $S4'^{\Delta}$ or $S4''^{\Delta}$ has a characteristic proof assignment.*

Proof. We will construct a characteristic proof assignment by induction on the principal labels. Let i be the smallest principal label of m-formulas in $\mathcal{F}$. If $\Box\phi^i$ is an R2-formula, the pair (ϕ, i) is assigned with a proof constant c and otherwise, the pair is assigned with an arbitrary proof term.

Now suppose that for any $k{<}i$, $p(\psi, k)$ has been determined. If $\Box\phi^i$ is initial, then following the previous step, assign a proof constant or an arbitrary proof term to (ϕ, i). If $\Box\phi^i$ has only one predecessor $\Box\psi^j$ and which is a δ-predecessor with $p(\psi, j){=}s$, let $p(\phi, i){=}s{+}t$ for a term t. For the other cases, let S be the set of proof terms consisting of $s\cdot t$ if $\langle\Box\psi^j, \Box\theta^k\rangle$ is a predecessor pair of $\Box\phi^i$ with $p(\psi, j){=}s$ and $p(\theta, k){=}t$, $!s$ if $\Box\psi^j$ is a γ-predecessor with $p(\psi, j){=}s$, s if $\Box\psi^j$ is an δ-predecessor with $p(\psi, j){=}s$, and a constant c if $\Box\phi^i$ is also an R2-formula. And then let $p(\phi, i) = \sum_{s\in S} s$. By this construction, $p(\phi, i)$ certainly satisfies all the equations in $\mathbf{E}_{\mathcal{F}}$ with $p(\phi, i)$ at the left-hand side. Continue this process until every pair is assigned with some term, then p is a characteristic proof assignment. $\dashv$

Theorem 5.6. *$\mathcal{H}$ is a proof in $GELP^-$, ELP^-, or ELP if and only if there is a proof $\mathcal{F}$ in $S4^{\Delta}$, $S4'^{\Delta}$, or $S4''^{\Delta}$ and a characteristic proof assignment p on $\mathcal{F}$ such that $\mathcal{H} = \mathcal{F}^p$.*

Proof. Given a proof $\mathcal{F}$ in $S4^{\Delta}$ [$S4'^{\Delta}$, $S4''^{\Delta}$] and a characteristic proof assignment p on $\mathcal{F}$, it's straightforward to check that $\mathcal{H} = \mathcal{F}^p$ is a proof in $GELP^-$ [ELP^-, ELP]. For the other direction, let $Tm(\mathcal{H})$ be the set of proof terms in $\mathcal{H}$, and $\eta\colon Tm(\mathcal{H}){\to}\mathbb{N}$ be an injective function linearizing the subterm relation on $Tm(\mathcal{H})$. Then for the translation induced by η such that $(t{:}\phi)^\eta = \Box(\phi^\eta)^{\eta(t)}$, $\mathcal{F} = \mathcal{H}^\eta$ is a proof in $S4^{\Delta}$ [$S4'^{\Delta}$, $S4''^{\Delta}$]. Now let p be the proof assignment such that for any $\Box\phi^i$ in $\mathcal{F}$, $p(\phi, i) = \eta^{-1}(i)$. p will satisfy all the equations in $\mathbf{E}_{\mathcal{F}}$ in which the R2-equations are derived from $\mathcal{H}$, and then p is a characteristic proof assignment p on $\mathcal{F}$ and $\mathcal{F}^p$ is $\mathcal{H}$. $\dashv$

Definition 5.7. *Given a formula or a sequence of formulas $\mathcal{D}$ in $L_\Box$, we call a label function $r\colon m(\mathcal{D}){\to}Tm$ a* realization *on $\mathcal{D}$.*

Each realization r induces a traslation from $\mathcal{D}$ to a sequence of formulas $\mathcal{D}^r$ in $L_:$ such that for any $x{\in}m(\mathcal{D})$, $\mathcal{D}^r(x){=}r(x){:}\mathcal{D}^r(x.\star)$.

Here is the proof realization result.

Theorem 5.8. *A proof $\mathcal{D}$ in S4, S4′, or S4″ is non-circular if and only if there exists a realization r such that $\mathcal{D}^r$ is a proof in GELP⁻, ELP⁻, or ELP, respectively.*

Proof. For the "only if" part, given a non-circular proof $\mathcal{D}$ in S4 [S4′, S4″], by Theorem 4.9, there is a numerical label function Δ on $\mathcal{D}$ such that $\mathcal{D}^{\Delta}$ is a proof in $S4^{\Delta}$ [$S4'^{\Delta}$, $S4''^{\Delta}$], and, by Theorem 5.6, there is a proof assignment p on $\mathcal{D}^{\Delta}$ such that $(\mathcal{D}^{\Delta})^p$ is a proof in $GELP^-$ [ELP^-, ELP]. Let r be the realization on $\mathcal{D}$ such that for any $x{\in}m(\mathcal{D})$, $r(x){=}p(\mathcal{D}^{\Delta}(x.\star), \Delta(x))$. Then $\mathcal{D}^r{=}(\mathcal{D}^{\Delta})^p$, so r is what we are looking for. For the other direction, let $\mathcal{H}{=}\mathcal{D}^r$ be a proof in $GELP^-$ [ELP^-, ELP]. Then, by Theorem 5.6, $\mathcal{H}{=}\mathcal{F}^p$ for some proof $\mathcal{F}$ in $S4^{\Delta}$ [$S4'^{\Delta}$, $S4''^{\Delta}$] and some proof assignment p. It can be checked that $\mathcal{D} = \mathcal{F}^{\Box}$. So $\mathcal{D}$ is non-circular.

In fact, for this direction, we can directly prove it from $\mathcal{H}$. If we disregard the superficial symbolic difference, language $L_{:}$ is one of labeled modal languages $L_{\mathbf{I}}$ with proof terms as labels. Then $\mathcal{D} = \mathcal{H}^{\Box}$ and $l_{\mathcal{H}} = r$. However r will be a non-circular proof label function on $\mathcal{D}$ since it needs to satisfy the conditions of proof terms on the modal axiom schemes. ⊣

Definition 5.9. *Given a proof $\mathcal{D}$ in S4, S4′, or S4″, a realization r on $\mathcal{D}$ is called* characteristic *if there is an increasing label function Δ on $\mathcal{D}$ and a characteristic proof assignmentp on $\mathcal{D}^{\Delta}$ such that $\mathcal{D}^r{=}(\mathcal{D}^{\Delta})^p$.*

Corollary 5.10. *$\mathcal{H}{=}\mathcal{D}^r$ is a proof in GELP⁻, ELP⁻, or ELP if and only if r is a characteristic realization on $\mathcal{D}$.*

The theorem realization result between S4-systems and LP-systems immediately follows.

Corollary 5.11. *An $L_{\Box}$ formula ϕ is a theorem of S4, S4′, or S4″ if and only if there is a realization r on ϕ such that ϕ^r is a theorem in GELP⁻, ELP⁻, or ELP, respectively.*

5.2 The Realization Theorem for LP

According to the algorithmic procedures we have so far, we are able to turn an S4G proof, to an $S4^{\Delta}G$ proof, to an $S4^{\Delta}$ proof, to an $S4'^{\Delta}$ proof, to an ELP^- proof, and hence an ELP proof, which is a supersystem of ELP^-. So given an S4 theorem ϕ, we are able to turn it into an ELP theorem ϕ^r. But if we want to realize an S4 theorem exactly to an LP theorem, we need one more step. One directly way is to find an algorithm converting an ELP proof to an LP proof by a translation on proof terms, but, to keep the flavor of this paper, we will determine the subclass of S4$''^{\Delta}$ proofs, and hence the subclass of non-circular S4$''$ proofs, called *stable*, which is precisely the class of proofs can be realized to LP proofs.

We call a set of m-formulas *C-stable* for C in { A1, A2, A4, R2 }, if every element of the set is so. A set is *stable* if it is C-stable for some C.

Definition 5.12. *Let S be a set of m-formulas in an S4$'^{\Delta}$ proof $\mathcal{F}$. $P_*(S)$ is the set of all $*$-predecessors of m-formulas in S for $* \in \{\alpha, \beta, \gamma, \delta\}$ (see Definition 5.2). We say an equivalence relation $\sim$ defined on m-formulas in $\mathcal{F}$ is* stable *if it satisfies the following conditions:*

1. *every induced equivalence class with respect to $\sim$ is stable,*
2. *given an equivalence class E, for each $* \in \{\alpha, \beta, \gamma\}$, $P_*(E)$ is a subset of some equivalence class, and $P_\delta(E)$ is a subset of a union of at most two equivalence classes,*
3. *all the equivalence classes together form a linear finite chain E_1, E_2, $\ldots E_n$ such that for any $\phi \in E_i$, $\psi \in E_j$ with ϕ a predecessor of ψ, $i < j$.*

A proof is stable *if we can define a stable equivalence relation on the proof.*

Lemma 5.13. *Every stable S4$''^{\Delta}$ proof $\mathcal{F}$ has a simple characteristic proof assignment.*

Proof. Let E_1, E_2, $\ldots E_n$ be a chain of equivalence classes of m-formulas in $\mathcal{F}$ such that if $\phi{\in}E_i$ is a predecessor of $\psi{\in}E_j$, then $i{<}j$. We will construct a simple characteristic proof assignment p such that for any m-formulas in the same equivalence class, the same proof term will be assigned. The construction is by induction on the index of the equivalence classes in the chain, and we will write $p(E_i){=}t$ to mean that the proof term t is assigned to all the m-formulas in E_i.

First of all, if there is an R2-formula in E_1, let $p(E_1){=}c$ for some proof constant c, otherwise $p(E_1){=}v$ for some proof variable v. Suppose that for any $i{<}k$, $p(E_i)$ is determined. If there is an A1-formula in E_k, then since the equivalence relation is stable, there are $i, j{<}k$ such that $P_\alpha(E_k){\subseteq}E_i$ and $P_\beta(E_k){\subseteq}E_j$. Let $p(E_k){=}p(E_i){\cdot}p(E_j)$. If there is an A2-formula in E_k, let $p(E_k){=}!p(E_i)$, provided $P_\gamma(E_k){\subseteq}E_i$. If there is an A4-formula in E_k, let $p(E_k){=}p(E_i){+}p(E_j)$, provided $P_\delta(E_k){\subseteq}E_i{\cup}E_j$ with $i{\leq}j$. If E_k is none of the above cases, then, following the procedure of dealing with E_1, assign a proof constant or proof variable to the m-formulas in E_k. One caveat here is that each time a new proof constant or a new proof variable is assigned.

This step will help to create a *normal realization*, which will be discussed at the end of this section. Continue this process until all equivalence classes are assigned, then p is a simple characteristic proof assignment of $\mathcal{F}$. $\dashv$

Theorem 5.14. *An S4$''^{\Delta}$ proof $\mathcal{F}$ is stable if and only if there exists a proof assignment p such that $\mathcal{F}^p$ is a proof in LP.*

Proof. If p is a simple characteristic proof assignment, then it is easy to see that $\mathcal{F}^p$ is an LP proof, and, by the previous lemma, such a proof assignment exists for a stable proof, and hence the "only if" part of the theorem is proved. For the other part, we define an equivalence relation on the set of m-formulas in $\mathcal{F}$ such that $\Box\phi^i{\sim}\Box\psi^j$ if and only if $p(\phi,i){=}p(\psi,j)$. Put the induced equivalence classes in order, E_1, E_2, $\ldots E_n$, such that $i{<}j$ if $p(E_i)$ is a subterm of $p(E_j)$ with $p(E)$ being the proof term assigned to all the m-formulas in E. Then each E_i is stable, since, if, say, there is an A1-formula in E_i, then all other m-formulas in E_i must not be an A2-, A4-, and R2-formula, because the application $\cdot$ is the main proof term operation of $p(E_i)$. Furthermore, $P_*(E_i){\subseteq}E_j$ for some $j{<}i$ with $*{\in}\{\,\alpha,\beta,\gamma\,\}$, and $P_\delta(E_i){\subseteq}E_j{\cup}E_k$ for some $j,k{<}i$. Therefore $\sim$, and hence $\mathcal{F}$, is stable. $\dashv$

Definition 5.15. *In an S4$''^{\Delta}$ proof, we call an A1-formula or A2-formula* standard*, if there is only one axiom in the whole proof in which the formula is the leading formula, and an A4-formula* standard *if there are at most two axioms in the whole proof in which the formula is the leading formula. An S4$''^{\Delta}$ proof is called* standard *if there is no non-standard formulas in the proof.*

Lemma 5.16. *Every standard S4$''^{\Delta}$ proof is stable.*

Proof. Basically, the identity relation between m-formulas is a stable equivalence relation. We can arrange the m-formulas in order based on their principal labels (the order of the formulas with the same principal labels does not matter), and then it can be checked that the identity relation satisfies all the conditions in Definition 5.12. $\dashv$

Here's the final step.

Proposition 5.17. *Every S4$''^{\Delta}$ proof can be extended to a stable proof.*

Proof. First of all, before extending the proof, if needed, the number labels will be modified. Notice that for any fixed numbers m, n, if we modify the number labels of the proof such that for any $k{>}m$, n is added to k, then the result is still an S4$''^{\Delta}$ proof (all the modal axioms are still modal axioms).

Hence we can always modify the proof such that the difference between two consecutive number labels is as wide as we would like it to be.

Let $\Box\phi^i$ be a non-standard m-formula in an S4$''^{\Delta}$ proof, and $\{\psi_j\}_{0\leq j\leq n}$ be the class of axioms in which $\Box\phi^i$ is the leading formula. We will add several formulas, including several axioms, into the proof such that for each ψ_j, it is no longer an axiom but a derived formula in the proof, and, although new axioms are included, no additional non-standard m-formulas are added to the proof. Therefore, after this procedure, the overall number of non-standard m-formulas in the proof is decreased. Continuing this process, a standard proof will be built.

The formulas we add to the proof are the following. First, we will add axioms ψ'_j into the proof, where ψ'_j is the axiom ψ_j with m-formula $\Box\phi^{k+2j}$ substituting for $\Box\phi^i$ as the leading formula of the axiom. The number k has to be carefully chosen such that $k{+}2n{<}i$, and for each j, $\Box\phi^{k+2j}$ is new to the proof and ψ'_j is still an axiom. This is the stage of the procedure where modification of the proof might be needed. We also add axioms $\Box\phi^k{\rightarrow}\Box\phi^{k+1}$, $\Box\phi^{k+2n-1}{\rightarrow}\Box\phi^i$, $\Box\phi^{k+2n}{\rightarrow}\Box\phi^i$, and $\Box\phi^{k+2j-1}{\rightarrow}\Box\phi^{k+2j+1}$ and $\Box\phi^{k+2j}{\rightarrow}\Box\phi^{k+2j+1}$ for $1{\leq}j{\leq}(n{-}1)$. Then, we add more formulas such that ψ_j are derived from these axioms. Furthermore, some of the axioms ψ_j might be applied by the rule of axiom necessitation in the original proof, but now they are not axioms. The last step is to add more formulas with the method discussed in Proposition 4.18 or 4.19 such that $\Box\psi_j$ is also derived in the extended proof. This completes the procedure. $\dashv$

Theorem 5.18 (Realization Theorem)**.**
A formula $\phi \in L_\Box$ is an S4 theorem if and only if there is a realization r such that ϕ^r is an LP theorem.

Proof. Given a S4G proof of the S4 theorem ϕ, according to Theorem, we can the proof into a S4G proof. Now following the algorithm given in Theorem and Theorem we can turn the S4G proof into a proof. Then by Theorem it is showed that such a S4G proof can be stablized. Finally, a proof realization algorithm given in is in order to turn such a stablized proof into an LP proof in which the conclusion is exactly what we want. $\dashv$

In [2], a special type of realization, called *normal realization*, is highlighted. It requires that the negative modal occurrences of an S4 theorem be realized to proof variables, and different occurrences to different variables. From the procedure we introduce here, it should be clear that initial modalities in a proof are set to be realized to variables. However, it is just the case that every negative m-formulas of an S4 theorem is initial, which can be witness by the fact that all the negative m-formulas of an S4 theorem can be assigned with the numerical label 0 when a cut-free Gentzen-style proof of the theorem is translated to an $S4^{\Delta}G$ proof. Furthermore, observing the Gentzen-style proof, we can actually assign different numerical labels to different negative m-formulas of the S4 theorem and hence translate them into different labeled m-formulas in the derived $S4^{\Delta}G$ proof. These negative labeled m-formulas will be kept to be initial when the $S4^{\Delta}G$ proof is turned into the standard $S4''^{\Delta}$ proof and hence realized to various proof variables.

6 Discussions

The two main topics of this paper are the completeness of non-circular S4 proofs, and the proof realization procedure from S4 to LP. The logical framework of $S4^{\Delta}$ is introduced to bridge the topics. All the technics and methods discussed here can be well adapted to concern with modal logics other than S4 without much difficulty, as long as cut-free Gentzen proof systems are available for these modal logics.[5]

The most straightforward attempt to prove the realization theorem is probably to try to show it by induction on the length of the Hilbert-style proofs of S4 theorems. But this attempt immediately have some difficulty when handling the case of *modus pones*. Given that both S4 theorems ϕ and $\phi \to \psi$ are realizable, meaning that there are realizations r_1 and r_2 such that ϕ^{r_1} and $(\phi \to \psi)^{r_2}$ are LP theorems, there is no clue as to how to unify r_1 and r_2 to form a realization r_3 such that both ϕ^{r_3} and $(\phi \to \psi)^{r_3}$ are LP theorems, and hence the *modus ponens* can be applied to get a realization for the S4 theorem ψ. This problem has been called for attention for quite some time.[6] Here, with our analysis, we can have an idea of the difficulty. For LP theorems ϕ^{r_1} and $(\phi \to \psi)^{r_2}$, we have their proofs $\mathcal{D}^{r_1}$ and $\mathcal{D}'^{r_2}$, in which, following our discussions here, $\mathcal{D}$ and $\mathcal{D}'$ are both non-circular Hilbert-style S4 proofs. Then the concatenation of sequences $\mathcal{D}\mathcal{D}'\psi$ forms a Hilbert-style proof of ψ. However the non-circularity of $\mathcal{D}$ and $\mathcal{D}'$ can't guarantee that the sequence $\mathcal{D}\mathcal{D}'\psi$ possesses the same quality and hence is realizable. Thus the question of unifying realizations of S4 theorems ϕ and $\phi \to \psi$ is just as finding an algorithm that can directly turn a circular proof to a non-circular one.

Finally, we would like to point out that although in our discussions, the circularity is attributed to proofs in modal logic, but the difficulty of

[5]For more discussions on this, see [19, 21].

[6]See [9, 11].

removing it is not substantially resting on the part of modal logic. It needs to take analysis on Hilbert-style proofs of classical tautologies. Take the circular proof presented at the end of Section 2 as an example:

$\phi_1 \equiv \Box(P \to Q) \to (\Box P \to \Box Q)$
$\phi_2 \equiv \Box(Q \to P) \to (\Box Q \to \Box P)$
$\phi_3 \equiv \Box(P \to Q) \to (\Box(Q \to P) \to (\Box P \to \Box P))$
$\phi_4 \equiv \Box(P \to Q) \to (\Box(Q \to P) \to (\Box Q \to \Box Q))$
$\phi_5 \equiv \Box(P \to Q) \to (\Box(Q \to P) \to ((\Box P \to \Box P) \wedge (\Box Q \to \Box Q)))$.

Notice that the intended proof in which this sequence is a fragment is that ϕ_3 is derived from ϕ_1 and ϕ_2 by eliminating the subformula $\Box Q$ and ϕ_4 is derived from ϕ_1 and ϕ_2 by eliminating $\Box P$. One quick response to this non-circularity is to add another axiom $\phi_* \equiv \Box(Q \to P) \to (\Box P \to \Box Q)$ to the proof such that ϕ_4, instead of being derived from ϕ_1 and ϕ_2, is derived from ϕ_2 and ϕ_* by eliminating the subformula $\Box P$ from these two axioms. Then it can be shown that the resulting proof is non-circular. However such a remedy can't always be applied. Suppose a fragment of a proof is as follows:

$\phi_1 \equiv \Box(P \to Q) \to (\Box P \to \Box Q)$
$\phi_2 \equiv \Box(Q \to P) \to (\Box Q \to \Box P)$
$\phi_3 \equiv \phi_1 \to (\phi_2 \to \phi_5)$
$\phi_4 \equiv \phi_2 \to \phi_5$
$\phi_5 \equiv \Box(P \to Q) \to (\Box(Q \to P) \to ((\Box P \to \Box P) \wedge (\Box Q \to \Box Q)))$,

where ϕ_4 is derived from ϕ_1 and ϕ_3, and ϕ_5 is derived from ϕ_2 and ϕ_4. Then the non-circularity of the proof is resting on the classical tautology

$$\begin{aligned}\phi_3 \equiv\, & (\Box(P \to Q) \to (\Box P \to \Box Q)) \\ & \to ((\Box(Q \to P) \to (\Box Q \to \Box P)) \\ & \quad \to (\Box(P \to Q) \to (\Box(Q \to P) \to ((\Box P \to \Box P) \wedge (\Box Q \to \Box Q))))).\end{aligned}$$

Therefore, an algorithm which can directly turn a circular proof into a non-circular one must be able to deal with the indirect steps of proofs that appeal to the classical tautologies of the above kind.

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