Temporalizing Modal Epistemic Logic

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Abstract

Timed Modal Epistemic Logic, tMEL, is a newly introduced logical framework for reasoning about the modeled agent's knowledge. The framework, derived from the study of Justification Logic, is adapted from the traditional Modal Epistemic Logic, MEL, to serve as a logically non-omniscient epistemic logic and dealing with problems where the the temporal constraint is an unavoidable factor. In this paper we will give a semantic proof for the formal connection between MEL and tMEL, the Temporalization Theorem, which states that every MEL theorem can be turned into a tMEL theorem if suitable time labels can be found for each knowledge statement involved in the MEL theorem. As a result, the proof also gives us a better understanding of the semantics on the both sides of the theorem.

1 Introduction

Contemporarily, the modal approach of epistemic logic, MEL, with its possible world semantics is the standard logical framework for reasoning about the mental qualities of agents [11, 8, 20]. But it is also well-noticed that the setting is defective. The modeled agents are able to know, say, all the logical consequences of their knowledge, a reasoning ability categorizing only ideal agents. Not surprisingly, approaches has been proposed to deal with the problem (e.g. [16, 13, 12, 7]). Based on the analysis that the problem is due to the agents' knowing too much, different apparatuses have been suggested to restrict the modeled agent's reasoning ability (*awareness function*, *impossible world*, *incomplete set of rules*, etc.). However, these restrictions have been argued to be ad hoc, and the rationality of the agency appears absent in these approaches [4]. Alternatively, it have been suggested that the problem can't be solved by proposing weaker logical frameworks, as what these approaches are trying to do. Instead, to solve the problem, a logical framework toward the epistemic foundation of agent theory should reveal the dynamic feature of agents's reasoning such that the propositions that are hard to know can be distinguished from the ones that are easy to.

Several logical setting can be counted as falling into this later group of approaches, which include Step Logic (later Active Logic) [6, 15], Algorithmic Logic [4, 5], Justification Logic [1, 2, 9], and timed Modal Epistemic Logic tMEL [17, 18], to name some of them. Among them, tMEL is distinct in the way that it is built up on the foundation of the original MEL logical framework, so keeping the flavor of possible world semantics, serves as a logically non-omniscient epistemic logic, so the modeled agents don't assume to possess the reasoning ability beyond the reach of human beings, and is capable of dealing with problems where deadline constrain is a factor, such as *The Nell & Dudley Problem* [14, 10, 3]. Roughly speaking, syntactically, each MEL formula of the form $\mathbf{K}\phi$ will be accompanied with a natural number i to form a formula $\mathbf{K}\phi^i$ in tMEL, purported to mean ϕ is known at the time i; and semantically, each world of a tMEL structure will be equipped with a syntactical device, called an awareness function, to capture when the agent is aware of a formula by the deductive procedure that the model designers assume the agent to possess. Then for example the MEL theorem $\mathbf{K}(\phi \rightarrow \psi) \rightarrow (\mathbf{K}\phi \rightarrow \mathbf{K}\psi)$, which can be interpreted as saying that the agent is able to perform *modus ponens*, has temporal counterparts in tMEL, $\mathbf{K}^i(\phi \rightarrow \psi) \rightarrow (\mathbf{K}^j\phi \rightarrow \mathbf{K}^k\psi)$, for numbers $i, j<k$, saying further that the rule takes time to apply. This temporalization of a MEL theorem into a tMEL theorem is in fact not an isolated result. One of the important metatheorems concerning tMEL, the *Temporalization Theorem*, making the formal connection between MEL and tMEL, states that every MEL theorem can be turned into a tMEL theorem if suitable time labels can be found for each knowledge statement (formula of the form $\mathbf{K}\phi$) involved in the MEL theorem.

The goal of this paper is to give a semantic proof of the result. Although a syntactic proof of the Temporalization Theorem has been given in the context of studying the proof relations between MEL, tMEL, and Justification Logic [19], a semantic proof of a logical result is always of its own interests. In particular, the generalization of the proof-theoretical method is restricted, for it needs to take cut-free Gentzen style proofs into account, and, as we can see, besides its promise of generalization, the semantic proof provided here shed light on both the tMEL and MEL semantics.

2 MEL and tMEL Logics

2.1 Modal Epistemic Logic

We begin with a presentation of the semantics, together with the axiom systems for reference, of the logics on the both side of the main result. We first review the possible world semantics, which is the foundation of tMEL semantics. The language of MEL is built up from a nonempty set of primitive propositions $\mathcal{P}$, boolean connectives, and a modal operator $\mathbf{K}$. To simplify the arguments, only cases of boolean connectives negation ($\sim$) and implication ($\rightarrow$) will be explicitly discussed. A well-formed MEL formula is defined according to the following grammar $\phi := p|{\sim}\phi|\phi{\rightarrow}\phi|\mathbf{K}\phi$, where $p \in \mathcal{P}$.

A *structure* or *a model* for MEL is a tuple $\langle W, R, \mathcal{V}\rangle$, where W is a set of worlds or epistemic alternatives, R is a binary relation, normally called accessible relation, defined on W, and $\mathcal{V}$ is a function assigning possible worlds to primitive propositions. The *satisfaction relation in a structure M* is recursively defined as follows:

$M, w \Vdash p \Leftrightarrow w \in \mathcal{V}(p)$,
$M, w \Vdash {\sim}\phi \Leftrightarrow M, w \nVdash \phi$,
$M, w \Vdash \phi \rightarrow \psi \Leftrightarrow M, w \nVdash \phi$ or $M, w \Vdash \psi$,
$M, w \Vdash \mathbf{K}\phi \Leftrightarrow M, w' \Vdash \phi$ for all $w' \in W$ with wRw'.

A formula is *valid in a structure* if it is satisfied in every world of the structure. Formulas which are valid in all structures compose the smallest MEL logic K, and its corresponding complete ans sound axiom system is:

Axioms
Classical propositional axiom schemes
$\mathbf{K}(\phi \rightarrow \psi) \rightarrow (\mathbf{K}\phi \rightarrow \mathbf{K}\psi)$
Inference Rules
if $\vdash \phi \rightarrow \psi$ and $\vdash \phi$, then $\vdash \psi$
if $\vdash \phi$, then $\vdash \mathbf{K}\phi$

Several extensions of K are often discussed in the literature. The following is a table of some well studied modal logical axioms, especially in the epistemic context, and their correpsponding conditions on the binary relation R:

	Axiom		R
T	$\mathbf{K}\phi\to\phi$	Truth Axiom	reflexive
4	$\mathbf{K}\phi\to\mathbf{K}(\mathbf{K}\phi)$	Positive Introspection Axiom	transitive
5	$\sim\mathbf{K}\phi\to\mathbf{K}(\sim\mathbf{K}\phi)$	Negative Introspection Axiom	euclidean

Let Λ be a subset of $\{T, 4, 5\}$. A $K\Lambda$-*structure* is a structure whose binary relation satisfies conditions corresponding to the axioms mentioned in Λ. We call a formula $K\Lambda$-*valid* if it is valid in all $K\Lambda$ structures. $K\Lambda$ *logic* contains all $K\Lambda$-valid formulas, and its complete and sound axiomatic counterpart is exactly the axiom system K plus axioms in Λ. For example $K45$ is the logic of all formulas valid in transitive and euclidean structures, and $K45$ axiom system is K plus axioms 4 and 5. Notice that the $KT4$ logic in our terminology is the familiar $S4$, and $KT45$ is $S5$. All the logics listed here are the targets of this paper. We will show altogether that theorems in these logics can be temporalized into their counterparts in tMEL logics.

2.2 Timed Modal Epistemic Logic

2.2.1 Semantics Basics

The language of tMEL is similar to the language of MEL except that the natural numbers are now part of the formula constructors. Natural Numbers are used to denote the passage of time, that is, in tMEL a simple structure of time, discrete, linear, with a beginning point, is considered. The grammar of well-formed tMEL formulas is: $\phi := p|{\sim}\phi|\phi\to\phi|\mathbf{K}^i\phi$, where $p \in \mathcal{P}$ and $i \in \mathbb{N}$ a natural number. $\mathbf{K}^i\phi$ is read as the formula ϕ is known at time i.

A *tMEL base* is a tuple $\mathcal{A} = \langle \mathbf{A}, \mathbf{f}\rangle$, where $\mathbf{A}$ is a set of tMEL formulas and $\mathbf{f}\colon \mathbf{A} \to \mathbb{N}$. Given a base $\mathcal{A} = \langle \mathbf{A}, \mathbf{f}\rangle$, we call a partial function α that associates tMEL formulas with natural numbers $\mathcal{A}$-*awareness function*, if it satisfies the following condition:

If $A \in \mathbf{A}$, then $\alpha(A) \leq f(A)$. (*Initial Condition*)

Furthermore, we will call an $\mathcal{A}$-awareness function *normal* if it satisfies two more conditions ($\alpha(\phi)\!\downarrow$ means $\alpha(\phi)$ is defined):

If $\alpha(\phi\to\psi)\downarrow$ and $\alpha(\phi)\downarrow$, then
$\alpha(\psi)\leq\max(\alpha(\phi\to\psi), \alpha(\phi))+1.$ (*Deduction by Modus Ponens*)

If $A \in \mathbf{A}$ and $f(A) \leq i$, then
$\alpha(\mathbf{K}^i A) \leq i + 1.$ (*Deduction by $\mathcal{A}$-Epistemization*)

Basically, an agent modeled by a tMEL logic is assumed to employ some kind of axiomatic method for reasoning, and the aim of an awareness function is to record the reasoning process of the modeled agent. Formulas in the set $\mathbf{A}$ of a base are supposed to be the formulas of which the truth are acceptable by the agent through non-deductive methods, such as perceiving some self-evident logical truths inherently or conveyed by others, and $\mathbf{f}$ indicate when these non-deductive methods takes place. Then those conditions for awareness function just reflect the rules that the agent can apply, and for an awareness function α, $\alpha(\phi) = i$ indicates that the first time when the agent accepts the truth of ϕ is i.

Given a base $\mathcal{A} = \langle \mathbf{A}, \mathbf{f} \rangle$, a tMEL $\mathcal{A}$-structure is a tuple $M = \langle W, R, \mathfrak{A}, \mathcal{V} \rangle$, where $\langle W, R, \mathcal{V} \rangle$ is a MEL structure and $\mathfrak{A} = \{\alpha_w\}$ is a collection of $\mathcal{A}$-awareness functions with one for each world $w \in W$. Then the *satisfaction relation in a tMEL structure M* the same as that in MEL structure except that the rule for modal formulas is changed to the following:

$M, w \Vdash \mathbf{K}^i\phi \Leftrightarrow M, w' \Vdash \phi$ for all $w' \in W$ with wRw',
and $\alpha_w(\phi) \leq i$.

It says that in the world $w \in W$ the agent knows a formula at the time i if and only if the formula is true in all possible worlds accessible from w and the agent accepts the truth of the formula before or equal to i.

A formula is *valid* in a tMEL structure if the formula is satisfied at all worlds in the structure. Given a base $\mathcal{A}$, a structure $M = \langle W, R, \mathfrak{A}, \mathcal{V} \rangle$ is a *$tK(\mathcal{A})$-structure* if $\mathfrak{A}$ consists of normal $\mathcal{A}$-awareness functions, and the *logic of $tK(\mathcal{A})$* is the set of formulas valid in all $tK(\mathcal{A})$-structures.

Similar to MEL logics, several extensions of $tK(\mathcal{A})$ are defined based on subclasses of $tK(\mathcal{A})$ structures. But now subclasses are determined not only by the binary relation R but also by the collection of awareness functions $\mathfrak{A}$, and its relation to the structure.

Given two awareness functions α, β, we write $\beta \leq \alpha$ to mean that $\beta(\phi) \leq \alpha(\phi)$ for every formula ϕ with $\alpha(\phi)\downarrow$. Let $M = \langle W, R, \mathfrak{A}, \mathcal{V} \rangle$ be a tMEL structure, and here are some more conditions on awareness functions:

If $\alpha_w(\phi) \leq i$, then $\alpha_w(\mathbf{K}^i\phi) \leq i+1$ (*Inner Positive Introspection*)
If $\alpha_w(\phi) \not\leq i$, then $\alpha_w(\mathbf{K}^i\phi) \not\leq i+1$ (*Inner Negative Introspection*)

For any wRw', $\alpha_{w'} \leq \alpha_w$ (Monotonicity)
For any wRw', $\alpha_w \leq \alpha_{w'}$ (Anti-Monotonicity)

If $M, w \Vdash \mathbf{K}^i\phi$, then $\alpha_w(\mathbf{K}^i\phi) \leq i+1$. (*Outer Positive Introspection*)
If $M, w \nVdash \mathbf{K}^i\phi$, then $\alpha_w(\mathbf{K}^i\phi) \not\leq i+1$. (*Outer Negative Introspection*)

Within a given structure, an awareness function is *positive regular (with respect to the structure)* if it satisfies the monotonicity and both inner and outer positive introspection, and *negative regular (with respect to the structure)* if it satisfies the anti-monotonicity and both inner and outer negative introspection. Some tMEL axioms and their corresponding conditions on the awareness functions in $\mathfrak{A}$ are listed in the following table:

	Axiom		$\mathfrak{A}$
tT	$\mathbf{K}^i\phi \rightarrow \phi$		none
$t4$	$\mathbf{K}^i\phi \rightarrow \mathbf{K}^j(\mathbf{K}^i\phi)$	$i < j$	positive regular
$t5$	$\sim\mathbf{K}^i\phi \rightarrow \mathbf{K}^j(\sim\mathbf{K}^i\phi)$	$i < j$	negative regular

Let Λ be a subset of $\{T, 4, 5\}$, and $\mathcal{A}$ be a base. A $tK(\mathcal{A})$-structure $\langle W, R, \mathfrak{A}, \mathcal{V}\rangle$ is a $tK\Lambda(\mathcal{A})$*-structure* if $\langle W, R, \mathcal{V}\rangle$ is a $K\Lambda$-structure and every awareness function in $\mathfrak{A}$ also satisfies the conditions corresponding the tMEL axioms in Λ. A formula is $tK\Lambda(\mathcal{A})$ *valid* if it is valid in all $tK\Lambda(\mathcal{A})$-structures. $tK\Lambda(\mathcal{A})$ *logic* contains all $tK\Lambda(\mathcal{A})$ valid formulas. So a $tK45(\mathcal{A})$ valid formula is valid in all $tK(\mathcal{A})$-structures whose binary relation is transitive and euclidean and its awareness functions are all both positive and negative regular.

2.2.2 Logical Bases and Axiomatization

Till now, there is no restriction on the base that is employed for the definition of tMEL semantics. But it will be more interesting if a base consists of *logical truths*, since it means the agent modeled by a tMEL logic with respect to the base have basic logical knowledge which will function like axioms in axiom systems in the agent's reasoning. Given bases $\mathcal{A} = \langle \mathbf{A}, \mathbf{f}\rangle$ and $\mathcal{B} = \langle \mathbf{B}, \mathbf{g}\rangle$, we will write $\mathcal{B} \subseteq \mathcal{A}$ to mean $\mathbf{B} \subseteq \mathbf{A}$ and $\mathbf{f}(B) \leq \mathbf{g}(B)$ for all $B \in \mathbf{B}$. A set of bases $\{\mathcal{A}_i (= \langle \mathbf{A}_i, \mathbf{f}_i\rangle)\}_{i\in\mathbb{N}}$ is an *ascending chain* if $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \ldots$, and a base $\mathcal{A}$ is the *limit* of the chain if $\mathcal{A} = \bigcup \mathcal{A}_i$, i.e., $\mathbf{A} = \bigcup \mathbf{A}_i$ and $\mathbf{f}(A) = \min\{\mathbf{f}_i(A) : \mathbf{f}_i(A)\downarrow\}$. The following is the definition of such bases:

Definition 2.1. *A base $\mathcal{A}$ is $tK\Lambda$-*logical *if one of following is true:*
(1) $\mathcal{A}$ is empty,
(2) $\mathcal{A}$ consists of $tK\Lambda(\mathcal{B})$-valid formulas with $\mathcal{B}$ $tK\Lambda$-logical,
(3) $\mathcal{A}$ is the limit of an ascending chain of $tK\Lambda$-logical bases $\{\mathcal{A}_i\}_{i\in\mathbb{N}}$ where $\mathcal{A}_{i+1}$ consists of $tK\Lambda(\mathcal{A}_i)$-valid formulas for every $i \in \mathbb{N}$.

Lemma 2.2. *If $\mathcal{A}= \langle \mathbf{A}, \mathbf{f} \rangle$ is a $tK\Lambda$ logical base, every formula in $\mathbf{A}$ is $tK\Lambda(\mathcal{A})$ valid.*

Given a tK-logical base $\mathcal{A}= \langle \mathbf{A}, \mathbf{f} \rangle$, there is a corresponding axiom system of the logic of $tK(\mathcal{A})$:

Axioms

Classical propositional axiom schemes

$\mathbf{K}^i(\phi \to \psi) \to (\mathbf{K}^j\phi \to \mathbf{K}^k\psi)$ $\quad i, j < k$ (*Deduction by Modus Ponens*)

$\mathbf{K}^iA \to \mathbf{K}^j(\mathbf{K}^iA)$ $\quad i < j$ if $A \in \mathbf{A}$ and $f(A) \le i$ (*Deduction by $\mathcal{A}$-Epistemization*)

$\mathbf{K}^i\phi \to \mathbf{K}^j\phi$ $\quad i < j$ (*Monotonicity*)

Inference Rules

if $\vdash \phi \to \psi$ and $\vdash \phi$, then $\vdash \psi$ (*Modus Ponens*)

if $A \in \mathbf{A}$ and $\mathbf{f}(A) \le i$, then $\vdash \mathbf{K}^iA$ (*$\mathcal{A}$-Epistemization*)

Let $\mathcal{A}$ be a $tK\Lambda$-logical base. For the logic of $tK\Lambda(\mathcal{A})$, the sound and complete corresponding axiom system is tK plus the tMEL axioms in Λ (more precisely, tK plus tX axioms with $X\in\Lambda$).

Theorem 2.3. *Given a a $tK\Lambda$-logical base $\mathcal{A}$, a tMEL formula ϕ is $tK\Lambda(\mathcal{A})$-valid if and only if it is provable in the $tK\Lambda(\mathcal{A})$ axiom system.*

So for a given $tK\Lambda$ logic, there is actually a collection of corresponding $tK\Lambda(\mathcal{A})$ logics introduced. The logical bases $\mathcal{A}$ are capturing the basic logical truths that agents are assumed to be aware of, and hence different $tK\Lambda(\mathcal{A})$ logics manifest different logical strengths. For example, if $\mathcal{A}$ is the empty base, then no formula of form $\mathbf{K}^i\phi$ is $tK\Lambda(\mathcal{A})$ valid. In [18], it is shown that there exists *comprehensive $tK\Lambda$-logical bases* $\mathcal{A}$ such that every $tK\Lambda(\mathcal{A})$ valid formula is in the base $\mathcal{A}$. One type of logical bases lying between the above two is of importance. A *full $tK\Lambda$-logical base* $\mathcal{A}$ is such that for any $tK\Lambda$ valid formula ϕ, though ϕ might not be in the base, $\mathbf{K}^i\phi$ is $tK\Lambda$ valid for some i. This feature of a full logical base is a desirable one, since it indicates that the agent modeled by a $tK\Lambda$ logic with respect to a full logical base has enough basic logical knowledge to derive to know all valid formulas.

Another good and natural feature that we would like a logical base to possess is *schematic*. By a *schematic logical bases* $\mathcal{A}= \langle \mathbf{A}, \mathbf{f} \rangle$, we mean that suppose $\phi \in \mathbf{A}$ and $\mathbf{f}(\phi) = i$, then if we add a fix number n to every number labels in ϕ to form a new tMEL formula ψ, then $\psi \in \mathbf{A}$ and $\mathbf{f}(\psi) = i$, too. This property suggests that the agent modeled by a $tK\Lambda$ logic with respect to a schematic logical bases is aware of the formula in $\mathbf{A}$ by schema, and hence for formulas falling into the same schema the agent is aware of them at the same time.

Fixing a $\Lambda \subseteq \{T, 4, 5\}$, our main goal is just to show that every $K\Lambda$ theorem can be *temporalized* to a theorem of a $tK\Lambda$ logic with respect to a schematic full logical base $\mathcal{A}$, that is, every $K\Lambda$ theorem can be turned into a $tK\Lambda(\mathcal{A})$ theorem by finding suitable number labels for knowledge statements involved in the $K\Lambda$ theorem, and equivalently, due to the completeness theorem, to show that every $K\Lambda$ valid formula can be temporalized to a $tK\Lambda(\mathcal{A})$ valid formula.

However, the route we take to prove the result will take several stages. First we need the following lemma proved in [18] (for simplication, all the logical bases are taken to be schematic in the following discussions):

Lemma 2.4. *A $tK\Lambda$-logical base $\mathcal{A}$ is full if and only if there is a comprehensive $tK\Lambda$-logical base $\mathcal{B}$ such that for every tMEL formula ϕ, ϕ is $tK\Lambda(\mathcal{A})$ valid if and only if ϕ is $tK\Lambda(\mathcal{B})$ valid.*

We call a logical base $\mathcal{A}= \langle \mathbf{A}, \mathbf{f} \rangle$ *principal* if $\mathbf{f}$ is the constant function $\mathbf{0}$, that is, $\mathbf{f}(\phi) = 0$ for all $\phi \in \mathbf{A}$. Let $\mathcal{A}$ be a principal comprehensive $K\Lambda$-logical base, and $\mathcal{B}$ an arbitrary comprehensive $K\Lambda$-logical base. In the next section we will show that a $K\Lambda$ valid formula can be temporalized to a $tK\Lambda(\mathcal{A})$ valid formula if and only if it can be temporalized to a $tK\Lambda(\mathcal{B})$ valid formula. This result together with the previous lemma shows that a $tK\Lambda$ logic with the principal comprehensive logical base and that with a full logical base have the same logical strength to temporalize $K\Lambda$ valid formulas. Then in the section after, we will prove that a $tK\Lambda$ logic with the principal comprehensive logical base indeed can temporalize every $K\Lambda$ valid formula to conclude our main result.

3 Comprehensive Bases

Before continuing, we need some notations and terminology for our discussions in this and the next section. For simplicity, we will use *subformulas* to mean *subformula occurrences* of a formula in this paper. According their positions in a formula, subformulas can be categorized either *positive* or *negative*: for a formula ϕ, ϕ is a positive subformula of itself and if $\theta \rightarrow \psi$, $\mathbf{K}\psi$ or $\sim\theta$ is a positive subformula, or if $\psi \rightarrow \theta$, $\mathbf{K}\theta$, or $\sim\psi$ is a negative subformula, ψ and θ are positive and negative subformula of ϕ respectively. Let ϕ be a MEL formula. We use $\mathcal{O}(\phi)$ to denote the set of all subformulas of ϕ with the form $\mathbf{K}\psi$, and $\mathcal{O}^+(\phi)$ and $\mathcal{O}^-(\phi)$ to denote the subsets of $\mathcal{O}(\phi)$ of positive and negative subformulas respectively. Given a function $\tau\colon \mathcal{O}(\phi) \rightarrow \mathbb{N}$, it will induce a natural translation, also denoted as τ, on ϕ such that ϕ^τ is a tMEL formula and τ fixes the primitive propositions, commutes with boolean connectives, and $(\mathbf{K}\psi)^\tau = \mathbf{K}^i(\psi^\tau)$ with $i = \tau(\mathbf{K}\psi)$. We will call $\tau\colon \mathcal{O}(\phi) \rightarrow \mathbb{N}$ a *temporalization function* on ϕ or a *t-function* on ϕ. For each MEL formula ϕ and a t-function τ on ϕ there is a corresponding tMEL formula ϕ^τ, and for each tMEL formula ψ there is a corresponding unique MEL formula ϕ (the resulting formula from removing number labels from ψ) and a unique t-function on ϕ such that $\psi = \phi^\tau$. So in the following we will simply write a tMEL formula as ϕ^τ with ϕ an MEL formula and τ a t-function on ϕ. Given a t-function τ on ϕ, $\tau + n$ is the t-function on ϕ such that $(\tau + n)(\mathbf{K}\psi) = \tau(\mathbf{K}\psi) + n$ for every subformula $\mathbf{K}\psi \in \mathcal{O}(\phi)$, and we will call $\phi^{\tau+n}$ the *n-shift* of ϕ^τ.

With these notions we can define a *schematic logical base* $\mathcal{A}(=\langle \mathbf{A}, \mathbf{f} \rangle)$ as that if $\phi^\tau \in \mathbf{A}$, $\phi^{\tau+n} \in \mathbf{A}$ for every n and $\mathbf{f}(\phi^{\tau+n}) = \mathbf{f}(\phi^\tau)$, and we can define a *temporalization* of a $K\Lambda$ valid formula ϕ as that there is a t-function τ on ϕ such that ϕ^τ is a $tK\Lambda$ valid formula with respect to a logical base.

We call a set S of tMEL formulas $tK\Lambda(\mathcal{A})$*-satisfiable* if all the formulas are satisfiable in a world of a $tK\Lambda(\mathcal{A})$ structure, and $tK\Lambda(\mathcal{A})$*-finitely satisfiable* if every finite subset of S is satisfiable. The compactness theorem holds for all $tK\Lambda(\mathcal{A})$ with $\mathcal{A}$ a logical base. The proof is basically by constructing the $tK\Lambda(\mathcal{A})$-structure $M = \langle W, R, \mathfrak{A}, \mathcal{V} \rangle$ composed of all maximal $tK\Lambda(\mathcal{A})$-finitely satisfiable set Γ of formulas, where $\Gamma R \Gamma'$ if and only if $\Gamma^\sharp \subseteq \Gamma'$ for $\Gamma^\sharp = \{\psi \mid \mathbf{K}\psi^i \in \Gamma\}$, and α_Γ and $\mathcal{V}$ are defined as $\alpha_\Gamma(\psi) = \min\{i \mid \mathbf{K}\psi^i \in \Gamma\}$ and $\mathcal{V}(P) = \{\Gamma \mid P \in \Gamma\}$, respectively. In the following discussions, this model will be referred to as the *canonical* $tK\Lambda(\mathcal{A})$ *model*. We will leave the qualification of this structure as a $tK\Lambda(\mathcal{A})$-structure and the *Truth Lemma*:

$M, \Gamma \Vdash \phi$ if and only if $\phi \in \Gamma$, for the readers to check.

Theorem 3.1 (Compactness Theorem)**.** *Given a $tK\Lambda$-logical base $\mathcal{A}$, a set of tMEL formulas is $tK\Lambda(\mathcal{A})$-satisfiable if and only if it is $tK\Lambda(\mathcal{A})$-finitely satisfiable.*

For a function $\mathbf{f} \colon \mathbf{A} \to \mathbb{N}$, we write $\mathbf{f}|_{\mathbf{B}}$ to mean the restriction of $\mathbf{f}$ to the subset $\mathbf{B}$ of $\mathbf{A}$.

Corollary 3.2. *Given a $tK\Lambda$-logical base $\mathcal{A} = \langle \mathbf{A}, \mathbf{f} \rangle$, a tMEL formula ϕ^τ is $tK\Lambda(\mathcal{A})$ valid if and only if there is a $\mathcal{A}' = \langle \mathbf{A}', \mathbf{f}' \rangle$ where $\mathbf{A}'$ is a finite subset of $\mathbf{A}$ and $\mathbf{f}'$ is $\mathbf{f}|_{\mathbf{A}'}$ such that ϕ^τ is $tK\Lambda(\mathcal{B})$ valid.*

We call an awareness function β is an *n-backshift* of an awareness function α providing for every ϕ if $\alpha(\phi^{\tau+n})\downarrow$, then $\beta(\phi^\tau)\downarrow$, and $\beta(\phi^\tau) = \max\{0, \alpha(\phi^{\tau+n}) - n\}$. A structure $M' = \langle W', R', \mathfrak{A}', \mathcal{V}' \rangle$ is an *n-backshift* of $M = \langle W, R, \mathfrak{A}, \mathcal{V} \rangle$ if $\langle W, R, \mathcal{V} \rangle = \langle W', R', \mathcal{V}' \rangle$ and every awareness function β_w in $\mathfrak{A}'$ is an n-backshift of α_w in $\mathfrak{A}$. We have the following lemma.

Lemma 3.3. *If a tMEL structure $M' = \langle W', R', \mathfrak{A}', \mathcal{V}' \rangle$ is an n-backshift of $M = \langle W, R, \mathfrak{A}, \mathcal{V} \rangle$, $M, w \vDash \phi^{\tau+n}$ if and only if $M', w \vDash \phi^\tau$ for any tMEL formulas ϕ^τ.*

The proof is simply by induction on the complexity of formulas. Finally, we also need the following:

Lemma 3.4. *For any $tK\Lambda$-logical bases $\mathcal{A}$, if ϕ^τ is $tK\Lambda(\mathcal{A})$ valid, then for any number n, $\phi^{\tau+n}$ is also $tK\Lambda(\mathcal{A})$ valid.*

The proof is basically by application of the Lemma 3.3. See Appendix. Here's the main result of this section.

Theorem 3.5. *A tMEL formula ϕ^τ is a valid formula of a $tK\Lambda$ logic with respect to a comprehensive logical base if and only if there is a t-function τ' on ϕ such that $\phi^{\tau'}$ is a valid formula of a $tK\Lambda$ logic with respect the principle comprehensive logical base.*

Check the proof in the Appendix.

4 Temporalization Theorem

So in the section we will complete the semantic proof of Temporalization Theorem. We first prove several interesting theorems about the tMEL semantics, which will lead us to the main result. In this section we fix a MEL logic $K\Lambda$ and its tMEL counterpart $tK\Lambda(\mathcal{A})$ with $\mathcal{A}$ the principal comprehensive logical base. All discussions will be relative to these fixed logics. We will omit the logic name. From the context it should be clear which logic ($K\Lambda$ or $tK\Lambda(\mathcal{A})$) is under discussion. We write $\vDash \phi^\tau$ to mean ϕ^τ is $tK\Lambda(\mathcal{A})$ valid. Notice that the most important feature of the principal comprehensive logical base is that if $\vDash \phi^\tau$ then $\vDash \mathbf{K}^0(\phi^\tau)$.

Definition 4.1. *Let ϕ be a MEL formula and τ and τ' two t-functions on ϕ,*
1. $\tau < \tau'$ if for any $\mathbf{K}\psi \in \mathcal{O}(\phi)$, $\tau(\mathbf{K}\psi) < \tau'(\mathbf{K}\psi)$.
2. $\tau \prec \tau'$ if for any $\mathbf{K}\psi \in \mathcal{O}^+(\phi)$, $\tau(\mathbf{K}\psi) < \tau'(\mathbf{K}\psi)$, and for any $\mathbf{K}\psi \in \mathcal{O}^-(\phi)$, $\tau'(\mathbf{K}\psi) < \tau(\mathbf{K}\psi)$

Lemma 4.2. *If $\tau \prec \tau'$ on ϕ, then $\vDash \phi^\tau \rightarrow \phi^{\tau'}$*

Proof. The proof is by induction on the complexity of formula ϕ. The base case is trivial. Suppose $\phi{=}{\sim}\psi$, then $\tau' \prec \tau$ on ψ, and by the Induction Hypothesis (IH), $\vDash \psi^{\tau'}{\rightarrow}\psi^\tau$, so $\vDash {\sim}(\psi^\tau){\rightarrow}{\sim}(\psi^{\tau'})$, and hence $\vDash ({\sim}\psi)^\tau{\rightarrow}({\sim}\psi^{\tau'})$. We skip to check the case for $\phi{=}\psi{\rightarrow}\theta$. Suppose $\phi{=}\mathbf{K}\psi$, then $\tau \prec \tau'$ on ψ. By IH, $\vDash \psi^\tau{\rightarrow}\psi^{\tau'}$. Since $\vDash \mathbf{K}(\psi^\tau{\rightarrow}\psi^{\tau'})^0$, $\vDash \mathbf{K}(\psi^\tau)^i{\rightarrow}\mathbf{K}(\psi^{\tau'})^j$ for $i < j$ and hence $\vDash (\mathbf{K}\psi)^\tau{\rightarrow}(\mathbf{K}\psi)^{\tau'}$. The case for implication is similar. This completes the proof. ⊣

Theorem 4.3. *Let ϕ be a MEL formula. If for every t-function τ, ϕ^τ is satisfiable, then the set S composed of formulas ϕ^τ for all τ is satisfiable.*

Proof. Suppose the set S is not satisfiable, then there is a finite subset $\{\phi^{\tau_1}, \ldots, \phi^{\tau_s}\}$ of S which is not satisfiable. So $\vDash {\sim}(\phi_1^{\tau_1} \wedge \ldots \wedge \phi_s^{\tau_s})$. By Lemma 3.4, for any $n \in \mathbb{N}$, $\vDash {\sim}(\phi_1^{\tau_1+n} \wedge \ldots \wedge \phi_s^{\tau_s+n})$. Hence we can pick a larger number n and a τ such that $\tau \prec \tau_i + n$ for each i. Then $\vDash {\sim}\phi^\tau$. A contradiction. So S is satisfiable. ⊣

Definition 4.4. *Let ϕ be an MEL formula.*
ϕ is t-satisfiable *if there is a τ on ϕ such that ϕ^τ is satisfiable, and* t-refutable *if there is a τ on ϕ such that ${\sim}\phi^\tau$ is satisfiable.*

ϕ is unboundedly *t-satisfiable (t-refutable) if there is a τ on ϕ such that for every $\tau' > \tau$ on ϕ there is a $\tau'' > \tau'$ on ϕ such that $\phi^{\tau''}$ is satisfiable (refutable).*

ϕ is upward-closedly *t-satisfiable (t-refutable) if there is a τ on ϕ such that for every $\tau' > \tau$, $\phi^{\tau'}$ is satisfiable (refutable).*

Definition 4.5. *A structure $M = \langle W, R, \mathfrak{A}, \mathcal{V}\rangle$ is* configurational *if for every w' with wRw', ϕ is unboundedly t-satisfiable at w' then $\mathbf{K}\phi$ is unboundedly t-satisfiable at w.*

Theorem 4.6. *Every satisfiable set is satisfied in a configurational structure.*

Proof. We will prove the canonical structure is configurational, and it is sufficient to prove the following lemma. Let $[\phi^\tau] = \{\phi^{\tau'} | \tau' > \tau\}$.

Lemma 4.7. *Let Γ be a maximal finitely-satisfiable set of tMEL formulas. If for a MEL formula $\sim\mathbf{K}\phi$ there is a τ on $\sim\mathbf{K}\phi$ such that $[(\sim\mathbf{K}\phi)^\tau] \subseteq \Gamma$, then $\Gamma^\sharp \cup [(\sim\phi)^{\tau'}]$ is satisfiable, for $\tau' > \tau$ on $\sim\phi$.*

Proof. In Appendix. ⊣

(*Proof of Theorem 4.6 Continued*) Suppose the canonical structure is not configurational, then there is a formula $\mathbf{K}\phi$, and a maximal finitely satisfiable set Γ such that $\mathbf{K}\phi$ is not unboundedly t-satisfiable at Γ but ϕ unboundedly t-satisfiable at all maximal finitely satisfiable set $\Gamma' \supseteq \Gamma^\sharp$. Then $\sim\mathbf{K}\phi$ is upward-closedly t-satisfiable at Γ, and hence there is a τ on $\sim\mathbf{K}\phi$ such that $[(\sim\mathbf{K}\phi)^\tau] \subseteq \Gamma$, by Truth Lemma. By the previous lemma, there is a $\tau' > \tau$ on $\sim\phi$ such that $\Gamma^\sharp \cup [(\sim\phi)^{\tau'}]$ is satisfiable. Then ϕ is not unboundedly t-satisfiable at some Γ'. A contradiction. So the canonical structure is configurational. ⊣

Given a $tK\Lambda$ structure $M = \langle W, R, \mathfrak{A}, \mathcal{V}\rangle$, we will call $\langle W, R, \mathcal{V}\rangle$ *the underlied $K\Lambda$ structure of M* and denote it as M°.

Theorem 4.8. *Let ϕ be an MEL formula, $M = \langle W, R, \mathfrak{A}, \mathcal{V}\rangle$ be a configurational structure and $w \in W$.*

1. If ϕ is upward-closedly t-satisfiable at (M, w) then ϕ is satisfiable at (M°, w),

2. If ϕ is upward-closedly t-refutable at (M, w) then ϕ is refutable at (M°, w), *Proof.* We will prove this by induction on the complexity of ϕ. The basic case is trivial. Suppose $\phi \equiv \sim\psi$ is upward-closedly t-satisfiable at (M, w), then ψ is upward-closedly t-refutable at (M, w). So ψ is refutable at (M°, w), and ϕ is satisfiable at (M°, w). The proof for the second condition is similar.

Suppose at (M, w), $\phi \equiv \psi \to \theta$ is upward-closedly t-satisfiable, we will show that either ψ is upward-closedly refutable or θ is upward-closedly satisfiable at (M, w). Suppose not, then for every τ on ψ there is a $\tau' > \tau$ on ψ such that ψ^τ is satisfiable, and for every τ on θ there is a $\tau' > \tau$ on θ such that $\theta^{\tau'}$ is refutable. Then for every τ on ϕ, we can always establish a $\tau' > \tau$ on ϕ such that $\psi^{\tau'}$ is satisfiable and $\theta^{\tau'}$ is refutable, and hence $\phi \equiv \psi \to \theta$ is not upward-closedly satisfiable. A contradiction. Then either ψ is upward-closedly refutable or θ is upward-closedly satisfiable at (M, w). So, by IH, ψ is refutable at (M, w) or θ is satisfiable at at (M, w), and hence $\psi \to \theta$ is satisfiable at (M, w). The other part for this case is straightforward. We skip the proof here.

Finally we deal with the modal case. Suppose $\phi \equiv \mathbf{K}\psi$, and ϕ is upward-closedly satisfiable at (M, w), then for every wRw', ψ is upward-closedly satisfiable at (M, w'), so ψ is satisfiable at (M°, w'), and hence $\mathbf{K}\psi$ is satisfied at (M°, w). Now we suppose ϕ is upward-closedly refutable at (M, w), then suppose for every wRw', ψ is unboundedly satisfiable at (M, w'), then $\mathbf{K}\psi$ is unboundedly satisfiable at (M, w), since M is configurational. Then $\mathbf{K}\psi$ is not upward-closedly refutable at (M, w). This contradicts to our assumption. So there is a w' with wRw' such that ψ is upward-closedly refutable at (M, w'), and hence ψ is not satisfiable at (M°, w'), so ϕ is not satisfiable at (M°, w). This completes the proof. ⊣

Let Λ be a subset of $\{T, 4, 5\}$. Here is our main results:

Theorem 4.9. *Given the principal comprehensive $tK\Lambda$-logical base $\mathcal{A}$, ϕ is $K\Lambda(\mathcal{A})$ valid if and only if there is a temporalization function τ on ϕ such that ϕ^τ is $tK\Lambda(\mathcal{A})$ valid.*

Proof. The direction from right to left is relatively trivial, please refer to the Appendix. We prove the other direction here by contraposition. Suppose there is no τ such that ϕ^τ is $tK\Lambda$-valid, then for all τ on ϕ, $\sim\phi^\tau$ is satisfiable. By Theorem 4.3, there is a structure M and a world w of the structure such that for all τ, $\sim\phi^\tau$ is satisfiable at (M, w). And then by Theorem 4.6, we can assume M is configurational. At last, by Theorem 4.8, $\sim\phi$ is satisfied at (M°, w). A contradiction. So the theorem is proved. ⊣

Corollary 4.10. *Given a full $tK\Lambda$-logical base $\mathcal{A}$, ϕ is $K\Lambda$ valid if and only if there is temporalization function τ on ϕ such that ϕ^τ is $tK\Lambda(\mathcal{A})$ valid.*

5 Conclusion

In this paper we render a semantic proof to the Temporalization Themorem, and the proof itself, which includes several lemmas and theorems, also gives us a closer look of the tMEL secmantics and its relation to the MEL semantics. Taken as an example, the Theorem 4.8 gives us the idea that the pattern of the truth-values of the formulas in the tail of such a sequence: $\{\phi^{\tau_i}\}_{i\in\mathbb{N}}$ with $\tau_i < \tau_{i+1}$ in a world of a tMEL structure is an indication of the truth value of ϕ in the same world of the underlied MEL structure. For the future work, we hope we can extend the method used here to provide a semantic proof for the Realization Theorem in Justification Logic, which is varied from tMEL in the way that, briefly, it is the proof terms, which enjoy a more complicated structure, that plays the roles in Justification Logic as that played by natural numbers in tMEL.

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Appendix

Some Proofs

Proof for Lemma 3.4. The syntactical proof of this lemma is simple (take a look at the axiom systems); however we give a semantic proof here to investigate the tMEL semantics and to render some skills and techniques that might be useful for the future work. Suppose that there is an n and a $tK\Lambda(\mathcal{A})$-structure $M = \langle W, R, \mathfrak{A}, \mathcal{V}\rangle$ such that $M, w \Vdash (\sim\phi)^{\tau+n}$. Let M' be the n-backshift of M, then by Lemma 3.3, $M', w \Vdash (\sim\phi)^{\tau}$. Now we have to prove that M' is also a $tK\Lambda(\mathcal{A})$-structure to finish the proof. That is, we have to show whatever conditions that are listed above are satisfied by $\alpha_w \in M$, are also satisfied by $\beta_w \in M'$. The proof is straightforward. We only check the case for the initial condition. Since $\phi^{\tau} \in \mathcal{A}$, $\phi^{\tau+n} \in \mathcal{A}$ (we suppose $\mathcal{A}$ is schematic). Then $\beta(\phi^{\tau}) = \max\{0, \alpha(\phi^{\tau+n}) - n\} \le f(\phi^{\tau+n}) = f(\phi^{\tau})$. So β_w is also an $\mathcal{A}$-awareness function. ⊣

Proof for Theorem 3.5. Given a logical base $\mathcal{A} = \langle \mathbf{A}, \mathbf{f}\rangle$, we first assign a *rank* to every $tK\Lambda(\mathcal{A})$ valid formula. Let S be the set of all $tK\Lambda(\mathcal{A})$ valid formulas, and $\emptyset$ be the empty logical base. If $\phi^{\tau} \in S$ is $tK\Lambda(\emptyset)$ valid, the rank is 1. Suppose we have assigned formulas in S with rank less than k. Let $\mathbf{B}$ be the set of formulas in $\mathbf{A}$ whose ranks less than k, and $\mathcal{B} = \langle \mathbf{B}, \mathbf{f}|_{\mathbf{B}}\rangle$. Then for formulas in S whose ranks are undefined and which are also $tK\Lambda(\mathcal{B})$ valid, their ranks are k. Now according to Corollary 3.2, every formula in S is of a finite rank. Since for a comprehensive logical base every valid formula of the $tK\Lambda$ logic with the base is in the base, so every formula in the base has a rank.

We first prove the *if-part* of the theorem. Let $\mathcal{A}(= \langle \mathbf{A}, 0\rangle)$ be the principal comprehensive $tK\Lambda$-logical base, and $\mathcal{B}(= \langle \mathbf{B}, \mathbf{g}\rangle)$ be a comprehensive $tK\Lambda$-logical base. We will actually show that if ϕ^{τ} is $tK\Lambda(\mathcal{A})$ valid, then there is a $n \in \mathbb{N}$ such that for any $m \ge n$, $\phi^{\tau+m}$ is $tK\Lambda(\mathcal{B})$ valid, by induction on the rank of $\phi^{\tau} \in \mathbf{A}$. When ϕ^{τ}'s rank is 1, which means ϕ is $tK\Lambda(\emptyset)$ valid, so ϕ is $tK\Lambda(\mathcal{B})$ valid. Then by Lemma 3.4, $\phi^{\tau+n}$ is $tK\Lambda(\mathcal{B})$ valid, too, for every n. So $\phi^{\tau+n} \in \mathcal{B}$. The base case holds. Now suppose ϕ^{τ}'s rank is $k>0$, then there is a finite base $\mathcal{A}' = \langle \mathbf{A}', 0\rangle$ such that ϕ^{τ} is $tK\Lambda(\mathcal{A}')$-valid, where $\mathbf{A}' = \{\phi_1^{\tau_1}, \ldots, \phi_s^{\tau_s}\}$ and for each i the rank of $\phi_i^{\tau_i}$ is less than k. Then by IH, there is an n_i for each i such that $\phi_i^{\tau_i+n_i}$ is $tK\Lambda(\mathcal{B})$-valid, and hence $\mathbf{g}(\phi_i^{\tau_i+n_i})\downarrow$.

Now picking m large enough such that $m > n_i$ and $m > \mathbf{g}(\phi_i^{\tau_i+n_i})$ for each i, we are going to show $\phi^{\tau+m}$ is $tK\Lambda(\mathcal{B})$-valid, and then finish the proof. Suppose $\phi^{\tau+m}$ is not $tK\Lambda(\mathcal{B})$-valid, $\sim\phi^{\tau+m}$ is satisfiable in a $tK\Lambda(\mathcal{B})$-structure M. Let M' be the m-backshift of M. Then $\sim\phi^{\tau}$ is satisfiable in M', by Lemma 3.3. So all we need to show is that M' is a $tK\Lambda(\mathcal{A}')$-structure. Everything is similar to the proof in the previous Lemma 3.4, except that we have to show that every m-backshift awareness function β_w in M' of the awareness function α_w in M is an $\mathcal{A}'$-awareness function. Since $\phi_i^{\tau_i+m}$ is in $\mathcal{B}$, $\alpha_w(\phi_i^{\tau_i+m})\downarrow$ and hence $\beta_w(\phi_i^{\tau_i})$ is defined. By the definition of m-backshift $\beta_w(\phi_i^{\tau_i}) = \max\{0, \alpha_w(\phi_i^{\tau_i+m}) - m\} = 0$. Hence every β_w in M' is $tK\Lambda(\mathcal{A}')$ valid and M' is a $tK\Lambda(\mathcal{A}')$-structure.

For the *only-if-part*, let $\mathcal{A}_i = \langle \mathbf{A}_i, 0\rangle$, where $\mathbf{A}_i = \{\phi^\tau \in \mathbf{A} \mid$ the rank of ϕ is equal to or less than $i\}$, and $\mathcal{B}_i = \langle \mathbf{B}_i, \mathbf{g}_i\rangle$, where $\mathbf{B}_i = \{\phi^\tau \in \mathbf{B} \mid$ the rank of ϕ is equal to or less than $i\}$ and $\mathbf{g}_i = \mathbf{g}|_{\mathbf{B_i}}$. We will prove that for every i, $\mathcal{B}_i \subseteq \mathcal{A}_i$. and hence $\mathcal{B} \subseteq \mathcal{A}$. We prove it by induction on the index. For the base case, both $\mathbf{A}_1$ and $\mathbf{B}_1$ are the collection of $tK\Lambda(\emptyset)$ valid formulas, so $\mathcal{B}_1 \subseteq \mathcal{A}_1$. Suppose $\mathcal{B}_i \subseteq \mathcal{A}_i$, then every $tK\Lambda(\mathcal{A}_i)$-awareness function is a $tK\Lambda(\mathcal{B}_i)$-awareness function, and hence every $tK\Lambda(\mathcal{A}_i)$-structure is a $tK\Lambda(\mathcal{B}_i)$-structure, so every $tK\Lambda(\mathcal{B}_i)$ valid formula is a $tK\Lambda(\mathcal{A}_i)$ valid formula. This completes the proof.

$\dashv$

Proof for Lemma 4.7. We will prove this theorem by contraposition. Suppose for an $\tau' > \tau$, $\Gamma^\sharp \cup [(\sim\phi)^{\tau'}]$ is not satisfiable. Then there are finitely many formulas $F_i \in \Gamma^\sharp$, and finitely many formulas $(\sim\phi_j)^{\tau_j} \in [(\sim\phi)^{\tau'}]$ such that the set $\{F_i\} \cup \{(\sim\phi_j)^{\tau_j}\}$ is not satisfiable, and hence $\vDash \sim((\bigwedge F_i) \wedge (\bigwedge (\sim\phi_j)^{\tau_j}))$ (i and j belong to some finite index sets which we do not mention here). It follows that $\vDash (\bigwedge F_i) \rightarrow (\bigvee \phi_j^{\tau_j})$. Define τ'' on ϕ such that for any $\mathbf{K}\psi \in O^+(\phi)$, $\tau'' = \max\{\tau_j(\mathbf{K}\psi)\} + 1$, and for any $\mathbf{K}\psi \in O^-(\phi)$, $\tau''(\mathbf{K}\psi) = \tau'(\mathbf{K}\psi)$. Then $\tau_j \prec \tau''$ and $\tau < \tau''$ on ϕ. Since $\vDash (\bigwedge F_i) \rightarrow (\bigvee \phi_j^{\tau_j})$ and $\vDash \phi^{\tau_j} \rightarrow \phi^{\tau''}$ for all j, then $\vDash (\bigwedge F_i) \rightarrow \phi^{\tau''}$. Since $F_i \in \Gamma^\sharp$, then $\bigwedge F_i \in \Gamma^\sharp$ and $\mathbf{K}^k(\bigwedge F_i) \in \Gamma$ for some k. Since $\vDash (\bigwedge F_i) \rightarrow \phi^{\tau''}$, then $\vDash \mathbf{K}^0((\bigwedge F_i) \rightarrow \phi^{\tau''})$, so $\vDash \mathbf{K}(\bigwedge F_i)^k \rightarrow \mathbf{K}(\phi^{\tau''})^l$, for $l > k$, and hence $\vDash \mathbf{K}(\phi^{\tau''})^l$. Then let τ''' on $\mathbf{K}\phi$ such that $\tau'''(\mathbf{K}\psi) = \tau''(\mathbf{K}\psi)$ for all $\mathbf{K}\psi \in O(\phi)$, and $\tau'''(\mathbf{K}\phi) =$

$\max\{\tau(\mathbf{K}\phi), l\}+1$, so $\tau''' > \tau$ on $\mathbf{K}\phi$ and $\models (\mathbf{K}\phi)^{\tau'''}$. It follows $(\mathbf{K}\phi)^{\tau'''} \in \Gamma$. So the lemma holds. $\dashv$

Proof for Theorem 4.9 Continued. We prove the direction from right to left here by contraposition. Suppose ϕ is satisfiable in a world w of a $K\Lambda$ structure $M = \langle W, R, \mathcal{V}\rangle$. Let $M^{+} = \langle W, R, \mathfrak{A}, \mathcal{V}\rangle$, where $\mathfrak{A}$ is the collection of awareness functions α_w such that $\alpha_w(\phi^{\tau}) = 0$ for all tMEL formulas ϕ^{τ}. Then all α_w are $tK\Lambda(\mathcal{A})$-awareness function, and hence M^{+} is a $tK\Lambda(\mathcal{A})$-structure. It can be check by induction then for any MEL formula ϕ and any t-function τ on ϕ, $M, w \Vdash \phi$ if and only if $M^{+}, w \Vdash \phi^{\tau}$ for any $w \in M$. This completes the proof. $\dashv$