On Incorporating Reasoning Time into Epistemic Logic

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Abstract

In this paper I argue for the naturality and importance of the concept of *reasoning-based knowledge*, in contrast with the more familiar *information-based knowledge*, which is interpreted plainly by the possible world semantic setting. In search of a formal system for the concept, I propose two extensions of the newly introduced epistemic logical framework, tMEL, which has formulas of form $K^i\phi$ to mean that ϕ is known by the agent at the time i. The two introduced extensions are $\mathsf{tMEL}^{\mathsf{K}}$ and tMEL^{∞}, with each of them having a device representing the information-based knowledge and reasoning-based knowledge respectively. These extensions will be investigated and compared, and then I will demonstrate an application of tMEL^{∞}, with a detour to discuss Moore's paradox from the reasoning-based knowledge point of view.

1 Introduction

Normally we trace the development of modern epistemic logic back to von Wright's writing in which we find the first axiomatization of epistemic logic. But what is rarely discussed about the work is that the epistemic concept formulated therein is actually "being verified." Or to put it in another way, von Wright had a definite epistemic interpretation when the system was laid down. "$\mathbf{V}\phi$," the primitive form of the proposed epistemic formulas, is suggested to read as ϕ *is known* or directly ϕ *is verified*.[1] Since von Wright didn't say much about the meaning of "being verified," we might understand it in its ordinary sense, i.e., to say a sentence *is known* is to say that there is a procedure for verifying the truth of the sentence. From the proposed axioms in the system, it is fair to say that the systematic natures that was observed by von Wright of the usage of the epistemic term and then formulated in the form of modal logic are that verification leads to the truth of the verified sentence, and that the body of human knowledge is closed under verification.[2]

The semantic of epistemic logic, whose story is relatively well-known, appeared late in Hintikka's work, and, in its modern form, is still a standard formal

[1] See [22].

[2] Ibid. The axiomatization is called M in von Wright's essay, which in today's nomenclature is the modal logical system T.

mechanism in present days. The idea behind the semantics is the old adage that "*knowledge is the elimination of uncertainty*" [10], and thus, according to the analysis, "a knows that p" means p is true in all the epistemic alternatives with respect to the knower or agent a's current situation; or in other words, a is in a position to be certain that p is the case. The original intent for the semantic study is, as Hintikka himself describes, to supplement what has been the endless example-counterexample type of debates on the adequacy of proposed syntactical epistemic principles at the time, with a concrete and even pictorial devise to work with [9]. From today's vantage point of view, Hintikka's project is of great success, especially given its enormous applications in the science of computing and artificial intelligence [5, 13]. But I still find the analysis unsatisfactory. Something is just missing in the picture. Basically, an epistemic event such as someone or some agent holding some knowledge only can be a *temporal event*. A known sentence can become unknown to an agent because the truth value of the sentence itself, like the state of weather, changes along with the change of the time, e.g. that a knows it is raining can't longer be true if it stops raining; or because the agent under discussion loses memory of what is originally known: I no longer know where my key is if I forgot it.

From a theoretical point of view, we have good reasons to focus our epistemic reasoning on scenarios in which the basic events to know about are general events, like that *the earth is round*, whose truth-value won't change at times, and the agents in discussion won't forget what have been known, as we might try to understand the behavior of an inanimate machine or concern with a committing agent who won't lose any relevant information when devoted to completing a task. But even so, we are beings able to enrich our life by expanding our knowledge, that means turning something unknown into known. We learn things from our contact with the external world, or getting information from others. But finally, even if we limit our attention to *close systems* where the agents are not supposed to communicate with the outside world, the agents can still increase their knowledge just by activating their inner reasoning faculty, to figure out something unknown previously.

Thus my complaint about Hintikka's semantic analysis is that it doesn't take *reasoning time* into account. The concept of knowing itself should not be understood only as the static condition of the truth-values of the known formula in the epistemic alternatives. It is also related to how the agent under discussion figuring out the truth of the known formula. Sentences such as "a knows that p," according to the analysis, involves some temporal measurement which tells us *when* the sentence p is known by the agent, and the magnitude of the measurement is reflecting the reasoning ability of the agent a and the difficulty of the sentence p to be reasoned out. This interpretation also accords with von Wright's original investigation; anyway, no matter concerning with the individual knowledge or the body of human being's knowledge, verification is a time-consuming procedure, which can't be found in Hintikka's analysis.

I call the concept of knowledge that is defined plainly by the possible-world semantics the *information-based knowledge*, and the concept I emphasize here the *reasoning-based knowledge.* In [17], I proposes an epistemic logical framework tMEL, *timed Modal Epistemic Logic*, in which the basic epistemic formulas have the form of $K^i\phi$ with the intended meaning that ϕ is known by the agent under discussion at the time i.[3] It is then a natural step to extend the systems in the framework with an epistemic connective to reflect the reasoning-based knowledge; and this is the main goal of this paper.

To achieve the goal, we will do several things in this paper. First in the next section, we will introduce the needed preliminary machineries, including the systems and semantics of MEL, Modal Epistemic Logic, and tMEL, which, for the expository purpose, will be presented in details. tMEL, just like MEL, is an umbrella name including many logics. To simplify the discussion, we will focus on $\mathsf{S4}$, the epistemic logic discussed in [8], and its corresponding counterpart $\mathsf{tS4}$ in tMEL. Then we introduce two logical frameworks extended from tMEL, $\mathsf{tMEL}^{\mathsf{K}}$ and tMEL^{∞}, with each of them including a device representing information-based knowledge and reasoning-based knowledge respectively. These frameworks will be introduced in Section 3 and 4, respectively, and, following that, a section is devoted to their comparison. Finally, in Section 6, we will utilize tMEL^{∞} to take some issue in epistemic logic which couldn't be handled by the plain possible world semantic setting; but before that a detour is made to discuss the Moore's paradox from the perspective of reasoning-based knowledge.

2 From MEL to tMEL

The formal languages we are going to see in this paper will all be extended from the language of propositional logic, built up from a set of propositional letters and a set of truth-functional connectives, which, for convenience, will only comprise connectives $\sim$ (*negation*) and $\rightarrow$ (*implication*). One single non-truth-functional modal connective will be added to these languages, which is denoted as K. Nonetheless, concerning with the modal connective, two types of modal formulas will be constructed. One is the *simple modal formulas*, constructed based on the rule that if ϕ is a formula, so is $K\phi$; and the other is the *labelled modal formulas*, based on the rule that if ϕ is a formula, so is $K^i\phi$ with i in a set of labels. Taken as examples, the *language of* MEL is extended from the language of propositional logic by the simple modal formulas and the *language of* tMEL is extended by labelled modal formulas with natural numbers as labels.

In the standard *possible world semantics*, a *structure* $M=\langle W, R, \mathcal{V}\rangle$ is composed of three components: a set of *possible worlds* W, a binary *accessibility relation*, $R \subseteq W \times W$, and a function $\mathcal{V}$ assigning to each propositional letter a set of possible worlds . In this paper we will only consider the modal logic $\mathsf{S4}$ and, in a way, its various variants, so the accessibility relation is taken to be *reflexive* and *transitive*, if not stated otherwise. But as you can see the method practiced here can be generalized to be concerned with other modal epistemic logics. The truth-values of MEL formulas is recursively given as follows:

$(M, w) \vDash p$ for a propositional letter p iff $w \in \mathcal{V}(p)$,,

$(M, w) \vDash \sim\phi \quad$ iff $\quad (M, w) \nvDash \phi$,

$(M, w) \vDash \phi \rightarrow \psi \quad$ iff $\quad (M, w) \nvDash \phi$ or $(M, w) \vDash \psi$,

[3]In [17], I have written the formulas as $K\phi^i$, instead of $K^i\phi$ to prevent the misinterpretation that treats the system as a multi-modal system with K^i as modalities for every i. But it has been suggested that the formulas of the original form are difficult to read, and hence now I adopt the form of $K^i\phi$.

$$(M, w) \Vdash K\phi \quad \text{iff} \quad (M, w') \Vdash \phi \text{ for all } w' \in W \text{ with } wRw'.$$

So $K\phi$ is suggested to read as "*it is known that* ϕ", or "*a knows that* ϕ" for the agent a. In this paper we will only discuss the case of a single agent. The corresponding complete axiom system of S4 is then the following:

Axioms:
A0 classical propositional axiom schemes,
A1 $K(\phi \to \psi) \to (K\phi \to K\psi)$,
A2 $K\phi \to K(K\phi)$,
A3 $K\phi \to \phi$,

Inference Rules
R1 if $\vdash \phi \to \psi$ and $\vdash \phi$, then $\vdash \psi$,
R2 if $\vdash \phi$, then $\vdash K\phi$.

The provability and validity for any logics discussed in this paper are treated in the usual way.

Next we consider the semantics of tMEL logics, which is augmented from the standard possible world semantics, and we will focus on the semantics of tS4.[4] One thing distinctive of tMEL logics is that they are *base-parameterized.* By a base, formally we mean a tuple $\mathcal{A} = \langle \mathbf{A}, \mathbf{f} \rangle$, with $\mathbf{A}$ a set of tMEL formulas, and $\mathbf{f}\colon \mathbf{A} \mapsto \mathbb{N}$ (natural numbers). We will explain the significance of bases later. Now given a base $\mathcal{A}$, an $\mathcal{A}$-*awareness function* α is defined as a partial function mapping tMEL formulas to natural numbers and satisfying the following conditions ($\alpha(\phi)\downarrow$ denotes $\alpha(\phi)$ is defined):

0. If $A \in \mathbf{A}$, then $\alpha(A) \leq \mathbf{f}(A)$.

(*Initial Condition*)

1. If $\alpha(\phi \to \psi)\downarrow$ and $\alpha(\phi)\downarrow$, then
$\alpha(\psi) \leq \max(\alpha(\phi \to \psi), \alpha(\phi)) + 1$.

(*Deduction by Modus Ponens*)

2. If $A \in \mathbf{A}$ and $f(A) \leq i$, then
$\alpha(K^i A) \leq i + 1$.

(*Deduction by $\mathcal{A}$-Epistemization*)

3. If $\alpha(\phi)\downarrow$ and $\alpha(\phi) \leq i$, then
$\alpha(K^i \phi) \leq i + 1$.

(*Inner Positive Introspection*)

The goal of these awareness functions is nothing but chronologically recording the deductive reasoning that the modeled agent practices, and these conditions reflect the logical rules that the depicted agent is assumed to perform, with each application of the rules taking one unit of time.[5] In order to do-

[4] tMEL is derived from the study of Justification Logic [2, 1]. For details, see [17].

[5] [4] might be the earliest work in which an awareness function is introduced. To call the functions defined here also *awareness function* is indeed inspired by their work, and from purely technical point view, the setting given here could be regarded as an extension of their work. But there is a significant ideological difference between our work and theirs. An awareness function as we deal with it here does not mean to tell us when the agent starts to recognize a formula, as if the agent wasn't aware of the formula before the time indicated by the awareness function; rather, the function employed here is telling us *at what time the agent first time notices the truth-value of the formula is possible to be true in relation to the formulas that s/he has deduced beforehand.*

cument the reasoning process of the agent, we need to assume the reasoning system employed by the agent. The simplest way to do that is to assume the agent is reasoning axiomatically. This assumption, which is not that realistic, might be improved by later research, but at this moment we rely on it. Then a *logical* base, whose formal definition will be given in the following, plays the role to give us the information of what axioms are used by the agent, and when at the latest s/he became aware of the truths of the axioms, since at the time the agent definitely will interact with a source of the information. Now corresponding to each condition of an $\mathcal{A}$-awareness function with $\mathcal{A}= \langle \mathbf{A}, \mathbf{f} \rangle$, the rule that is supposed to be able to be performed by the agent is the following:

0. *Initial Condition:* This condition says nothing about any special kind of deduction rule to be performed. It indicates that at the very latest the agent will be aware of an axiom A employed by the agent at that time $\mathbf{f}(A)$.

1. *Deduction by Modus Ponens*: in one unit of time, the agent is able to perform *modus ponens.*

2. *Deduction by $\mathcal{A}$-Epistemization*: in one unit of time, the agent is able to perform $\mathcal{A}$-Epistemization, which means to deduce the agent's knowledge of an axiom that is employed by the agent.

3. *Inner Positive Introspection*: in one unit of time, the agent is able to perform some kind of introspection in order to be in a state of awareness of that, right before the time of the performance, the agent knows the formula which s/he has been aware of.

Notice that these conditions are for a tS4 awareness function. Obviously, the condition **2** is subsumed under the condition **3**. The reason for this separation is for the consideration of generality; the condition **3** is typical of a tS4 agent, not of an agent with weaker reasoning ability, and the condition **2** is something we want to reserve for the weaker agents.[6]

Fix a base $\mathcal{A}$. By a tS4($\mathcal{A}$)-*structure* we mean a tuple $M= \langle W, R, \mathfrak{A}, \mathcal{V} \rangle$, where $\langle W, R, \mathcal{V} \rangle$ is an S4 structure and $\mathfrak{A}$ is a collection of $\mathcal{A}$-awareness functions α_w, one for each world $w \in W$, satisfying the *monotonicity condition*[7]: given a tMEL formula ϕ, if $\alpha_w(\phi) \downarrow$, then $\alpha_{w'}(\phi) \leq \alpha_w(\phi)$. Informally, the condition, which is not needed for weaker tMEL logics, is stating that a tS4 agent can only imagine a formula known equally or earlier in time at the accessible world, which means tS4 agents are so firm about their knowledge that in the possibilities, only new sources of information can make them have the knowledge early. So the basic idea behind this semantics is that at each world, there is a static world view, like before, which includes the truth-values of those basic events, represented by propositional letters, and the reasoning process taken by the agent at the world.

Now we can define the truth-value for tMEL formulas. For the propositional cases, they are treated in the same way as those of MEL formulas, and for a labelled modal formulas, its truth-value is given as follows:

$$(M, w) \vDash K^i\phi \quad \text{iff} \quad (M, w') \vDash \phi \text{ for all } w' \in W \text{ with } wRw', \text{ and } \alpha_w(\phi) \leq i.$$

According to this analysis, an agent knows at the time i that ϕ is the case if and only if it is true at all accessible possible worlds and the agent is able to,

[6] See [17] for these weaker logics.

[7] Don't confuse this condition with the *monotonicity axiom* that we will see later.

by activating her/his deductive reasoning faculty on the information the agent has at the world, be aware of the truth of the formula before that time.

To define a tS4 *logical base*, we need some efforts, since the definition of the logical truths, i.e. the valid formulas, will depend on the base to be worked with; however, we also need every formula in the logical base to be logical truths. So we start from the empty base, which means the agent at the beginning is not assumed as being aware of any logical truth. But still with this base we can deduce some interesting logical truths, which include the classical tautologies and the ones reflect the conditions in the definition of awareness functions, including axiom instances of the following systems. But no formulas of the form $K^i\phi$ are logical truths; that is, an agent has no logical knowledge with empty base.

Some terminology is in order. Given bases $\mathcal{A}=\langle\mathbf{A}, f\rangle$ and $\mathcal{B}=\langle\mathbf{B}, g\rangle$, $\mathcal{B}\subseteq\mathcal{A}$ means $\mathbf{B}\subseteq\mathbf{A}$ and $f(B)\leq g(B)$ for all $B\in\mathbf{B}$. A collection of bases $\{\mathcal{A}_i(=\langle\mathbf{A}_i, f_i\rangle)\}_{i\in\mathbb{N}}$ is an *ascending chain* if $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \ldots$, and a base $\mathcal{A}$ is the *limit* of the chain if $\mathcal{A} = \bigcup \mathcal{A}_i$, i.e., $\mathbf{A} = \bigcup \mathbf{A}_i$ and $f(A) = \min\{f_i(A) : f_i(A)\downarrow\}$.

Definition 2.1. *A base $\mathcal{A}$ is* tS4 logical *if one of the following is true:*
(1) $\mathcal{A}$ is empty,
(2) in $\mathcal{A}=\langle\mathbf{A}, f\rangle$, $\mathbf{A}$ consists of tS4*($\mathcal{B}$) valid formulas with $\mathcal{B}$ a* tS4 *logical base,*
(3) $\mathcal{A}$ is the limit of an ascending chain of tS4 *logical bases $\{\mathcal{A}_i\}_{i\in\mathbb{N}}$, with $\mathcal{A}_{i+1}$ consisting of* tS4$(\mathcal{A}_i)$ *valid formulas.*

Lemma 2.2. *If $\mathcal{A}=\langle\mathbf{A}, \mathbf{f}\rangle$ is a tS4 logical base, every formula in $\mathbf{A}$ is tS4($\mathcal{A}$) valid.*

Let $\mathcal{A}$ be a tS4 logical base. The axiom system of tS4($\mathcal{A}$) is as follows:

Axioms:
A0 Classical propositional axiom schemes,
A1 $K^i(\phi \to \psi) \to (K^j\phi \to K^k\psi) \quad i, j < k$,
A1' $K^iA \to K^j(K^iA) \quad i < j$ if $A \in \mathbf{A}$ and $f(A) \leq i$,
A2 $K^i\phi\to K^j(K^i\phi) \quad i < j$,
A3 $K^i\phi \to \phi$,
A4 $K^i\phi \to K^j\phi \quad i < j$, (*Monotonicity Axiom*)

Inference Rules
R1 if $\vdash \phi \to \psi$ and $\vdash \phi$, then $\vdash \psi$,
R2 if $A \in \mathbf{A}$ and $f(A) \leq i$, then $\vdash K^iA$, (*$\mathcal{A}$-Epistemization*)

for all $i, j, k \in \mathbb{N}$.

Theorem 2.3 (Theorem 4.5 in [17])**.** *Given a* tS4 *logical base $\mathcal{A}$, a* tMEL *formula is* tS4*($\mathcal{A}$) valid if and only if it is provable in the axiom system of* tS4*($\mathcal{A}$).*

So corresponding to an MEL logic, such as S4, there is in fact a family of tMEL logics, tS4($\mathcal{A}$), introduced, with each tS4 logical base $\mathcal{A}$ to indicate the basic logical truths to be used by the agent. We have seen the empty base. We can also have a *comprehensive logical base* which include all valid formulas.[8] One logical base especially interests us. We call a tMEL logic, also the logics introduced later on, has the *internalization property* if for any valid formula

[8]If $\mathcal{A}$ is the limit of an ascending chain of tS4 logical bases $\{\mathcal{A}_i\}_{i\in\mathbb{N}}$, with $\mathcal{A}_{i+1}$ consisting of all tS4($\mathcal{A}_i$) valid formulas, then $\mathcal{A}$ is comprehensive. For the details, see [17].

ϕ, there is an $i{\in}\mathbb{N}$ such that $K^i\phi$ is valid. Then we call a logical base $\mathcal{A}$ *full*, if, using the base, the tMEL logic, as well as the logics defined later, have the internalization property. Certainly a comprehensive base is full, but so is a base containing all the axiom instances of the schemes listed in the above system. Given the completeness result, this can be proved by induction on a proof of the valid formula, with only axioms A1 and A1' applied.[9] We can understand agents with full or stronger logical bases as having enough basic logical truths from which they can have *complete* epistemic logical knowledge about themselves. Logics with full or stronger logical bases normally have nice properties, which include the following one concerning the formal relation between MEL logics and their tMEL counterparts. Here's the version for S4 and tS4:

Theorem 2.4 (Temporalization Theorem, Corollary 4.10 of [18])**.** *Given a full* tS4 *logical base $\mathcal{A}$, an* MEL *formula ϕ is a* S4 *valid formula if and only if there is a suitable label for each of the epistemic modalities K in ϕ such that turning all the epistemic modalities in ϕ into modalities with their suitable labels K^i, we have an* tS4$(\mathcal{A})$ *valid formula.*[10]

3 Logic of tS4$^{\mathsf{K}}$

The first logical framework we are going to introduce is rather a direct combination of MEL and tMEL. *The language for* tMEL$^{\mathsf{K}}$, and hence *for* tS4$^{\mathsf{K}}$, is extended from the language of tMEL by having the simple modal formulas of the form $K\phi$.

A tMEL$^{\mathsf{K}}$ base is a tuple $\mathcal{A}{=}\langle\mathbf{A},\mathbf{f}\rangle$, with $\mathbf{A}$ a set of tMEL$^{\mathsf{K}}$ formulas, and $\mathbf{f}\colon\mathbf{A}\mapsto\mathbb{N}$. Given a tMEL$^{\mathsf{K}}$ base $\mathcal{A}$, $M{=}\langle W,R,R_t,\mathfrak{A},\mathcal{V}\rangle$ is a tS4$^{\mathsf{K}}(\mathcal{A})$-*structure*, if $R\subseteq R_t$, $\langle W,R,\mathcal{V}\rangle$ is an S4-structure, and $\langle W,R_t,\mathfrak{A},\mathcal{V}\rangle$ is a tS4$(\mathcal{A})$-structure with the domains of the awareness functions α_w in $\mathfrak{A}$ consisting of tMEL$^{\mathsf{K}}$ formulas, instead of just tMEL formulas. The truth-values of the tMEL$^{\mathsf{K}}$ formulas for the modal cases are defined as follows:

$$(M,w)\vDash K\phi \quad\text{iff}\quad (M,w')\vDash\phi \text{ for all } w'\in W \text{ with } wRw',$$

and for a labelled modal formula:

$$(M,w)\vDash K^i\phi \quad\text{iff}\quad (M,w')\vDash\phi \text{ for all } w'\in W \text{ with } wR_tw', \text{ and } \alpha_w(\phi)\leq i.$$

Then a tS4$^{\mathsf{K}}$ *logical base* is recursively defined in the same way as the one for tS4, except that it is defined on the language of tMEL$^{\mathsf{K}}$. Given a tS4$^{\mathsf{K}}$ logical base $\mathcal{A}$, the axiom system tS4$^{\mathsf{K}}(\mathcal{A})$ is the following:

I.

Axioms:
A0 Classical propositional axiom schemes,
A1 $K^i(\phi\to\psi)\to(K^j\phi\to K^k\psi)\quad i,j<k$,
A1' $K^iA\to K^j(K^iA)\quad i<j$ if $A\in\mathbf{A}$ and $f(A)\leq i$,
A2 $K^i\phi{\to}K^j(K^i\phi)\quad i<j$,
A3 $K^i\phi\to\phi$,
A4 $K^i\phi\to K^j\phi\quad i<j\quad K^i\phi\to K^j\phi\quad i<j$,

[9] This is the reason we keep the condition **2** of awareness function for weaker tMEL logics.
[10] A constructive proof of this theorem can be found in [19].

Inference Rules
R1 if $\vdash \phi \to \psi$ and $\vdash \phi$, then $\vdash \psi$,
R2 if $A \in \mathbf{A}$ and $f(A) \le i$, then $\vdash K^i A$, ($\mathcal{A}$-*Epistemization*)

II.

Axioms:
A0 classical propositional axiom schemes
A1 $K(\phi \to \psi) \to (K\phi \to K\psi)$,
A2 $K\phi \to K(K\phi)$,
A3 $K\phi \to \phi$,

Inference Rules
R1 if $\vdash \phi \to \psi$ and $\vdash \phi$, then $\vdash \psi$,
R2 if $\vdash \phi$, then $\vdash K\phi$, (*Necessitation Rule*)

III. (Connection Axiom)

A1 $K^i\phi \to K\phi$, for any i.

The system, as it is presented, is composed of three parts, which explicitly shows that it is a combination of the systems S4 and tS4, though some of the axioms and rules are listed repeatedly. The only exception is *Connection Axiom*, which reflects the idea that the agent can only reason to know what is determined by the static possible world setting. The semantic condition which corresponds to this axiom is that $R \subseteq R_t$. On the contrary, reversing the relation, we can't obtain any connection theorem. Observe that even though we can infer from $K\phi$ is true in a world that ϕ is true in all R_t-related worlds, given that R_t included in R, it is not guaranteed that the agent is aware of the truth of ϕ in the world.

The discussed condition can also be set to $R = R_t$, with the resulting semantics still validating the same set of formulas. But to keep them separated, we can introduce more of this type of tMEL-MEL combination logics, e.g. the tS4+S5 logic, whose semantics is the same as what is introduced here, except that R needs to satisfy the equivalence relation, and whose axiom system is the same as the one listed here but with an additional 5 axiom for the modality K. Although this type of logics are not the focus of this paper, but they have their own theoretical interest worth further examination.

Theorem 3.1. *Given a* $\mathsf{tS4}^{\mathsf{K}}$ *logical base* $\mathcal{A}$*, a* $\mathsf{tMEL}^{\mathsf{K}}$ *formula is* $\mathsf{tS4}^{\mathsf{K}}(\mathcal{A})$ *valid if and only if it is* $\mathsf{tS4}^{\mathsf{K}}(\mathcal{A})$ *provable.*

Proof. The completeness results for these logics can be proved based on the standard canonical model method. We sketch the proof. For any $\mathsf{tS4}^{\mathsf{K}}(\mathcal{A})$ maximal consistent set Γ, let α_Γ be the partial function such that for any $\mathsf{tMEL}^{\mathsf{K}}$ formulas ϕ, $\alpha_\Gamma(\phi) = i$ if and only if $K^i\phi{\in}\Gamma$, and for $j{<}i$, $K^j\phi{\notin}\Gamma$, and $\Gamma^\# {=} \{\phi | \Box\phi \in \Gamma\}$. Let $M{=}\langle W, R, R_t, \mathfrak{A}, \mathcal{V}\rangle$ be the canonical model, in which W is the set of all $\mathsf{tS4}^{\mathsf{K}}(\mathcal{A})$ maximal consistent sets, $R{\subseteq}W{\times}W$ such that $\Gamma R \Gamma'$ if and only if $\Gamma^\# {\subseteq} \Gamma'$, $R_t = R$, $\mathfrak{A}{=}\{\alpha_\Gamma\}$ for $\Gamma{\in}W$, and $\Gamma{\in}\mathcal{V}(p)$ if and only if $p \in \Gamma$, for a propositional letter p. Then it can be proved that this is a $\mathsf{tS4}^{\mathsf{K}}(\mathcal{A})$-structure and Truth Lemma holds for M. $\square$

4 Logic of $\mathsf{tS4}^\infty$

In this section we introduce a system extended from $\mathsf{tS4}$ by having a device representing the reasoning-based concept of knowledge. This can be done in a similar way as we did above by introducing an epistemic modality K into $\mathsf{tS4}$ and endowing it with the reason-based knowledge interpretation. But since in $\mathsf{tS4}$ we have the *Monotonicity Axiom*: $K^i\phi \to K^j\phi$ for $i < j$, we will take advantage of it. If an agent won't forget what s/he knows, then to say the agent knows something *at some time* is the same as describing the agent as knowing that *eventually*. Hence, instead of introducing a new modality, we will extend the label set from natural numbers $\mathbb{N}$ to extended natural numbers $\mathbb{N}^\infty$ and regard formulas of the form $K^\infty\phi$ as the ones we expect. By doing so, as we will see, the axiom systems can be simplified.

Now, formally the language of tMEL^∞ is extended from the language of tMEL by adopting the extended natural numbers $\mathbb{N}^\infty$ as the labels for the labelled modal formulas, where the order of ∞ is that $x{<}\infty$ for any $x{\in}\mathbb{N}$ and $\infty{\leq}\infty$. A tMEL^∞ base is a tuple $\mathcal{A}{=}\langle \mathbf{A}, \mathbf{f}\rangle$, with $\mathbf{A}$ a set of tMEL^∞ formulas, and $\mathbf{f}\colon \mathbf{A} \mapsto \mathbb{N}$ (noticing that it is $\mathbb{N}$, not $\mathbb{N}^\infty$). Given a tMEL^∞ base $\mathcal{A}$, a $\mathsf{tS4}^\infty(\mathcal{A})$-*structure* is a $\mathsf{tS4}$ *structure* $M{=}\langle W, R, \mathfrak{A}, \mathcal{V}\rangle$, with the language of tMEL^∞ as the domain of awareness functions $\alpha_w{\in}\mathfrak{A}$.

The conditions for the truth-values of $\mathsf{tS4}^\infty$ formulas are exactly the same as the conditions for those of $\mathsf{tS4}$ formulas, including the following:

$$(M, w) \vDash K^i\phi \ \text{ iff } \ (M, w') \vDash \phi \text{ for all } w' \in W \text{ with } wRw', \text{ and } \alpha_w(\phi) \leq i.$$

The labelled modal formula $K^\infty\phi$ gets its truth-value through this condition, and notice that the range of the awareness functions for the tMEL^∞ semantics is the same as the range of the functions for the tMEL semantics, which is $\mathbb{N}$, and hence whenever $K^\infty\phi$ is true at some world then there is a $K^i\phi$ true at the world with $i \in \mathbb{N}$.

A $\mathsf{tS4}^\infty$ *logical base* is recursively defined in the same way as the one for $\mathsf{tS4}$, except that it is defined on the language of $\mathsf{tS4}^\infty$. Given a $\mathsf{tS4}^\infty$ logical base $\mathcal{A}$, the axiom system of $\mathsf{tS4}^\infty(\mathcal{A})$ is the following:

Axioms:
A0 Classical propositional axiom schemes,
A1 $K^i(\phi \to \psi) \to (K^j\phi \to K^k\psi) \quad i, j < k$,
A1' $K^iA \to K^j(K^iA) \quad i < j$ if $A \in \mathbf{A}$ and $f(A) \leq i$,
A2 $K^i\phi{\to}K^j(K^i\phi) \quad i < j$,
A3 $K^i\phi \to \phi$,
A4 $K^i\phi \to K^j\phi \quad i < j$, (*Monotonicity Axiom*)

Inference Rules
R1 if $\vdash \phi \to \psi$ and $\vdash \phi$, then $\vdash \psi$,
R2 if $A \in \mathbf{A}$ and $f(A) \leq i$, then $\vdash K^iA$,

for all $i, j, k \in \mathbb{N}^\infty$.

These systems are almost the same as the axiom systems of $\mathsf{tS4}$, except that the label variables i, j, k range over $\mathbb{N}^\infty$ instead of $\mathbb{N}$. But exactly because of this exception, more axioms are permitted by this system. For example, $K^5(\phi \to \psi) \to (K^7\phi \to K^\infty\psi)$ and $K^\infty(\phi \to \psi) \to (K^\infty\phi \to K^\infty\psi)$ are both instances of the axiom A1, and, corresponding to Connection Axiom in the $\mathsf{tS4}^K$ system, $K^i\phi \to K^\infty\phi$ is an instance of A4 axiom.

To proceed to prove the completeness theorem for $\mathsf{tS4}^\infty$ logics, one should notice that the compactness theorem does not hold for these systems, and hence the canonical model method can't be used. Consider the following sequence of formulas $\sim K^0\phi, \sim K^1\phi, \ldots, K^\infty\phi$. This is a consistent sequence and finitely satisfiable but not satisfiable, for if $K^\infty\phi$ is true at a world of a structure, then there must be some formulas $K^i\phi$ with finite i also true at the world. So alternatively we will prove this theorem by constructing a structure based on the set of relative maximal consistent sets. And it turns out, as you can see, the method is general enough to be adapted to give alternative proofs for the completeness theorem of $\mathsf{tS4}$ logics.

Theorem 4.1. *Given a* $\mathsf{tS4}^\infty$ *logical base* $\mathcal{A}$*, a* $\mathsf{tS4}^\infty$ *formula is* $\mathsf{tS4}^\infty(\mathcal{A})$ *valid if and only if it is* $\mathsf{tS4}^\infty$ *provable.*

Proof. For a tMEL^∞ formula ϕ, let $\mathrm{Sub}^+(\phi)$ be the set of all subformulas of ϕ and their negations. Given a $\mathsf{tS4}^\infty$ logical base $\mathcal{A}=\langle \mathbf{A}, \mathbf{f}\rangle$ and a tMEL formula ϕ, we say a set $\Gamma \subseteq \mathrm{Sub}^+(\phi)$ is a $\mathsf{tS4}^\infty(\mathcal{A})$ maximal consistent set relative to ϕ if 1) it is consistent, that is, from Γ we won't prove contradiction, and 2) for any formula ψ in $\mathrm{Sub}^+(\phi)$ but not in Γ, $\Gamma \cup \{\psi\}$ is inconsistent. For a set of tMEL^∞ formulas Γ, we define $\Gamma^\sharp=\{\phi \mid K^i\phi \in \Gamma, \text{and } i{\in}\mathbb{N}^\infty\}$, $\Gamma^\sharp_{\mathrm{Fin}}=\{\phi \mid K^i\phi \in \Gamma, \text{and } i \in \mathbb{N}\}$, and $\kappa_\Gamma=\max\{i \in \mathbb{N} \mid \sim K^i\phi \in \Gamma\}$. Now we inductively create a set of tMEL^∞ formulas U relative to Γ, stage by stage. At stage 0, $\psi \in U$ if $\psi \in \mathbf{A}$ and $\mathbf{f}(A) = 0$ or $K^0\psi \in \Gamma$. At the stage $n{\neq}0$, we put formula ψ into U, if ψ is not in U and satisfies one of the following conditions:

1. if $\psi{=}A \in \mathbf{A}$, $\mathbf{f}(A) = n$.
2. if there are formulas $\theta \to \psi$, θ in U,
3. if $\psi{=}K^{n-1}A$, where $A \in \mathbf{A}$, $\mathbf{f}(A) \le n-1$.
4. if $\psi{=}K^{n-1}\phi$, where $\phi \in U$,
5. if ψ is in $\Gamma^\sharp_{\mathrm{Fin}}$, and $n = \min\{i \mid K^i\psi \in \Gamma\}$,
6. if ψ is in $\Gamma^\sharp$ but not in $\Gamma^\sharp_{\mathrm{Fin}}$, and $n = \kappa_\Gamma + 1$.

Let W be the collection of relative maximal consistent sets, R be the binary relation such that for any Γ, Γ' in W, $\Gamma R \Gamma'$ if and only if $\Gamma^\sharp \subseteq \Gamma'$, $\mathcal{V}$ maps propositional letters to subsets of W such that $\Gamma \in \mathcal{V}(p)$ if and only if $p \in \Gamma$ for a propositional letter p, and $\alpha_\Gamma(\phi) = i$ if and only if ϕ is added to the set U_Γ at the ith stage, and let $M=\langle W, R, \mathfrak{A}, \mathcal{V}\rangle$. Then it can be checked that this is a $\mathsf{tS4}^\infty(\mathcal{A})$ -structure and the Truth Lemma: for every formula $\psi \in \mathrm{Sub}^+(\phi)$, $M, \Gamma \models \psi$ if and only if $\psi \in \Gamma$, holds. Now if ϕ is consistent then ϕ will belong to some relative maximal consistent set and hence it is satisfiable. We leave the reader to check that M is a $\mathsf{tS4}^\infty(\mathcal{A})$- structure. Here we prove the Truth Lemma for the case of labelled modal formulas. Suppose $K^i\psi \in \mathrm{Sub}^+(\phi)$ and $M, \Gamma \models K^i\psi$ for $i \in \mathbb{N}$, then $\psi \in U_\Gamma$ at a stage earlier than or equal to i. But if $\sim K^i\psi \in \Gamma$, then ψ can't be added into U_Γ before or equal to i, a contradiction, so $K^i\psi \in \Gamma$. The case for $K^\infty\psi$ is similar, since $M, \Gamma \models K^\infty\psi$, then $M, \Gamma \models K^i\psi$ for some i. For the other direction, suppose $K^i\psi$ is in Γ for $i \in \mathbb{N}^\infty$, then ψ is in Γ' for all $\Gamma R \Gamma'$, and $\alpha_\Gamma(\psi) \le i$, for $i \in \mathbb{N}$, and $\alpha_\Gamma(\psi) \le j$ for some $j \in \mathbb{N}$ when $i = \infty$, so $M, w \models K^i\psi$. □

5 Comparisons

Now we undertake to compare the logics $\mathsf{tS4}^K$ and $\mathsf{tS4}^\infty$. We will compare them in the way of putting formulas of the form $K\phi$ in $\mathsf{tS4}^K$ and $K^\infty\phi$ in $\mathsf{tS4}^\infty$ on a par,[11] since they are the formulas introduced to represent information-based knowledge and reasoning-based knowledge respectively. The main difference we try to capture is that for an $\mathsf{tS4}^\infty$ agent, whether $K^\infty\phi$ is true to him or not completely depend on the reasoning ability of the agent, i.e. in formal term, the logical base of the system describing the epistemic strength of the agent. But this is not always the case for $K\phi$ in $\mathsf{tS4}^K$.

Lemma 5.1. *For any $\mathsf{tS4}^\infty$ logical base $\mathcal{A}$, if formula $K^\infty\phi$ is $\mathsf{tS4}^\infty(\mathcal{A})$ valid, then there is an $i \in \mathbb{N}$ such that $K^i\phi$ is $\mathsf{tS4}^\infty(\mathcal{A})$ valid.*

Proof. Given a $\mathsf{tS4}^\infty$ logical base $\mathcal{A}$, we can create an $\mathsf{tS4}^\infty(\mathcal{A})$ awareness function $\alpha^*_\mathcal{A}$ such that for any $\mathsf{tS4}^\infty(\mathcal{A})$ awareness function α, if $\alpha^*_\mathcal{A}(\phi)\downarrow$ then $\alpha(\phi) \leq \alpha^*_\mathcal{A}(\phi)$. The construction of $\alpha^*_\mathcal{A}$ is similar to the construction of α_Γ in the above proof. We will inductively create a set of tMEL^∞ formulas U stage by stage. At stage 0, $\psi \in U$ if $\psi \in \mathbf{A}$ and $\mathbf{f}(A) = 0$. At the stage $n{\neq}0$, we put formula ψ into U, if ψ has not been in U yet and satisfies one of the following conditions:

1. if $\psi{=}A \in \mathbf{A}$, $\mathbf{f}(A) = n$.
2. if there are formulas $\theta \to \psi$, θ in U,
3. if $\psi{=}K^{n-1}A$, where $A \in \mathbf{A}$, $\mathbf{f}(A) \leq n-1$.
4. if $\psi{=}K^{n-1}\phi$, where $\phi \in U$.

Let $\alpha^*_\mathcal{A}$ to be the partial function such that for any tMEL^∞ formula ϕ, $\alpha^*_\mathcal{A}(\phi)\downarrow$ if $\phi \in U$, and $\alpha^*_\mathcal{A}(\phi){=}i$ if ϕ is added into U at the stage i. Then it is not difficult to show that this awareness function satisfies our requirement, including that if $\alpha^*_\mathcal{A}(\phi)\downarrow$ then $\alpha(\phi) \leq \alpha^*_\mathcal{A}(\phi)$, for any $\mathsf{tS4}^\infty(\mathcal{A})$ awareness function α. Now suppose $K^\infty\phi$ is $\mathsf{tS4}^\infty(\mathcal{A})$ valid, then $\alpha^*_\mathcal{A}(\phi)$ must be defined. So $K^i\phi$ is $\mathsf{tS4}^\infty(\mathcal{A})$ valid, if $\alpha^*_\mathcal{A}(\phi) \leq i$. □

But this is not generally true for $\mathsf{tS4}^K$ logics.

Lemma 5.2. *There is a $\mathsf{tS4}^K$ logical base $\mathcal{A}$, such that $K\phi$ is $\mathsf{tS4}^K(\mathcal{A})$ valid, but no $i \in \mathbb{N}$ such that $K^i\phi$ is $\mathsf{tS4}^K(\mathcal{A})$ valid.*

Proof. Consider the extreme case to take $\mathcal{A}$ as the empty set. Then for a classical tautology ϕ, $K\phi$ is $\mathsf{tS4}^K(\mathcal{A})$ valid, but no formula of the form $K^i\phi$ is logical valid. □

So in $\mathsf{tS4}^K$, the formulas $K\phi$ like the final goals, waiting for the agent who has enough reasoning ability to reach. In the literature, it is sometime called the *implicit knowledge* of the agent [11]. But on the other hand, the concept of knowledge captured by $K^\infty\phi$ is, with any assumption of the reasoning strength of the agent, what is actually known by the agent, but *with the omission of the*

[11] This is kind of a misleading way to put it, since ϕ in $K\phi$ as a formula in $\mathsf{tS4}^K$, and ϕ in $K^\infty\phi$ as a formula in $\mathsf{tS4}^\infty$ cannot always mean the same formula. But for convenience, and for comparison, we will still write them as such, and readers should remember they do not have to mean the same formula.

exact time of knowing. Of course if we assume that the agent has enough reasoning ability, then things will become different. Consider the situation within $\mathsf{tS4}^K$ logics:

Lemma 5.3. *Given a full* $\mathsf{tS4}^K$ *logical base* $\mathcal{A}$*, such that* $K\phi$ *is* $\mathsf{tS4}^K(\mathcal{A})$ *valid, then there is an* $i \in \mathbb{N}$ *such that* $K^i\phi$ *is* $\mathsf{tS4}^K(\mathcal{A})$ *valid.*

Proof. Since $K\phi$ is $\mathsf{tS4}^K(\mathcal{A})$ valid, so is ϕ. Then since $\mathcal{A}$ is full, there is an $i \in \mathbb{N}$ such that $K^i\phi$ is $\mathsf{tS4}^K(\mathcal{A})$ valid. □

Now consider the situation between logics. If in $\mathsf{tS4}^\infty(\mathcal{A})$, $\mathcal{A}$ is supposed to be full, then in a sense $K^\infty\phi$ is no difference with the $K\phi$ in $\mathsf{tS4}^K(\mathcal{A})$, and also the $K\phi$ in $\mathsf{S4}$. Let $(\ )^\infty$ be the function mapping tMEL^K formulas to tMEL^∞ formulas by the following recursive conditions:

$(p)^\infty = p$ if p is a propositional variable,
$(\sim \phi)^\infty = \sim (\phi)^\infty$,
$(\phi \to \psi)^\infty = (\phi)^\circ \to (\psi)^\infty$,
$(K^i\phi)^\infty = K^\infty(\phi)^\infty$ for any an $i \in \mathbb{N}$,
$(K\phi)^\infty = K^\infty(\phi)^\infty$.

Theorem 5.4. *Let* ϕ *be an* MEL *formula,* $\mathcal{A}$ *a full* $\mathsf{tS4}^\infty$ *logical base, and* $\mathcal{B}$ *any* $\mathsf{tS4}^K$ *logical base.*
(1) ϕ *is* $\mathsf{S4}$ *valid if and only if* ϕ *is* $\mathsf{tS4}^K(\mathcal{B})$ *valid;*
(2) ϕ *is* $\mathsf{tS4}^K(\mathcal{B})$ *valid if and only if* $(\phi)^\infty$ *is* $\mathsf{tS4}^\infty(\mathcal{A})$ *valid.*

Proof. The easiest way to prove it is by comparing the proofs in the different proof systems. Here we demonstrate the direction from left to right of (2). Suppose ϕ has a $\mathsf{tS4}^K(\mathcal{B})$ proof, then it is easy to check by induction that for every ψ in the proof, $(\psi)^\infty$ is either a $\mathsf{tS4}^\infty(\mathcal{A})$ axiom or $\mathsf{tS4}^\infty(\mathcal{A})$ theorem. For the case that either $K\psi$ is derived by necessitation, or $K^j\psi$ is derived by $\mathcal{B}$-Epistemization, from ψ in $\mathsf{tS4}^K(\mathcal{B})$, since $(\psi)^\infty$ is provable in $\mathsf{tS4}^\infty(\mathcal{A})$, and $\mathcal{A}$ a full logical base, so there is an $i \in \mathbb{N}$ such that $K^i(\psi)^\infty$ is provable, and hence $(K\psi)^\infty$, or $(K^j\psi)^\infty$, is $\mathsf{tS4}^\infty(\mathcal{A})$ provable. □

6 Some Discussions

In this final section, I will use an example to demonstrate that tMEL^∞ could give us insights into problems that could not be brought about by epistemic logics plainly equipped with the possible world semantics. But before that, we will deviate from the formal spirit of this paper to discuss a more pragmatic issue with regard to the approach of our analysis of knowledge. As what has been said in the beginning of the paper, a knowledge event is temporal, hence moving out of the logical territory to decide the truth-value of an utterance of a knowledge claim, we need to determine what's the time the event denoted by the knowledge claim is supposed to take place. That is, when people say "*a knows that* ...," or "*a does not know that* ...," to determine the truth-values of these utterances, we need to, first, from the context, determine when is the time of the event that the utterer refers to. Normally it is the time of the utterance, but that's not mandatory; and for the purpose of discussion, we don't take it for granted.[12] The pragmatic issue we are going to discuss is Moore's paradox.

6.1 Moore's Paradox

Moore's paradox has attracting many discussions. An anthology has been published on the issue [7]. Moore has an opinion on it, so is Wittgenstein , and Wittgenstein even claims that the discovery of the paradox is the most important philosophical contribution of Moore [12]. So if someone says to us that "*it's raining, and I don't know it's raining*," we sense some kind of absurdity.[13] But what is said is not against any semantic rules, and it is quite possible to be true. Then what's the source of the sense of absurdity? Roughly speaking, Moore thought it's our asserting implying our knowledge about what is asserted, and the one who utters the Moore's sentence makes what is implied by the assertion, knowing that it's raining, go against what is asserted, not knowing that it's raining [14, 3]. On the other hand, Wittgenstein thought the sense of absurdity indicates that in everyday practice, our first personal claim about our own inner state of mind is a way of speaking about something outside us. Thus the second part of an utterance of a Moorean sentence is saying something of the world, i*t's not raining*, which is against what is said in the first part of the sentence, *it's raining* [20, 21].

I only very briefly demonstrate their ideas, which might miss some refined points, and it is properly better to understand these analyses as being about belief, rather than knowledge.[14] But my object here is to give the idea of what a possible answer to the paradox could be. There is no judgement about whose analysis is better, and my personally attitude to a paradox as interesting as this, is that we can get different insights from our different analyses about it; and in this paper I try to offer another one.

So to determine the truth-value of an utterance such as that *"p and I don't know that p,"* we have to determine what's the time of the non-knowing event that the utterer refers to. It has long been known, if what the utterer intends to say is *"p and I didn't know that p* in the past*,"* i.e., the intended time of the part of the no-knowledge claim of the utterance is earlier than the time of the utterance, then there is no absurdity in the case.[15] It's a normal utterance, with the content that could be true and could be false. Now what if what the utterer tries to say is about the future, such as *"p and I will not know that p* forever*"*? In literature, as far as I know, there is no discussion about this case

[12]Of course, in English, the verb form of a sentence indicates its tense. So no problem to judge if the utterer is saying something in English about, e.g., the past. But for the purpose of argument, one could suppose the utterer speaks in some language in which the tense is not built into the grammar of sentences, such as Chinese. Our conclusion does not rely on this supposition.

[13]The most discussed Moore's paradox is the belief version that *"it's raining, and I don't believe it's raining*," but Moore himself did discuss the knowledge version, which is in [15], published after his death.

[14]Since our main results here focus on knowledge, I try to keep all the discussions uniformly about knowledge. But it needs not to be so. Hence if you like, you could change the discussions in this part to be about belief. Of course knowledge and belief have some essential differences which could make the discussions of knowledge version and belief version of Moore's paradox go in different directions. However, as you can see, our analysis here is not dependent on these differences.

[15]See [3, 8].

yet, but I think there is no suspense in this one – it is simply false. There is no sense of absurdity arising either, if this is what the utterer means. The difference is that this time the utterer just says something false. But we can continue to ponder what if what the utter tries to say is, within the Moorean utterance, that *"I won't know p* tomorrow*"*? *"*six hours later*"*? or *"*10 minutes later*"*? Does the utterer say something absurd? I would say "no" to these cases. Then there must be a case in which the content of the utterance is false, and the intended time of the non-knowing claim is the earliest among such cases. Let's call that time as the *t-time*, *the turning time*, of the Moorean utterance. We would say the t-time in our raining example is, of course, not in the past, but very very close to the time of the utterance. My claim is that it is this *closeness* that is the source of absurdity. Back to the normal circumstance. If no further information is provided or needed to be concerned, a Moorean utterance is normally interpreted as meaning that *"it's raining, and I don't know it's raining* now*"*; but the closeness makes the truth-value of the content of the utterance in a shaky place.

To make my point clearer, consider the following example. If what the utterer says is that *"p and I don't know q,"* with q a logical consequence of p but very far away in the sense that it takes a long derivation to get q from p. Then do we still sense any absurdity? To judge that the utterer says something definitely false, we probably understand what the utterer tries to mean is that *"p and I don't know q* forever*."* The t-time for this utterance is far in the future. Hence for most cases we will just understand the utterance as a normal utterance. On the other hand, consider this: *"I don't know what the weather is, but as a matter of fact it is raining."* Absurd! So if my analysis is correct, then the source of the sense of the absurdity of the utterance of Moorean sentence is in a way just like the absurdity we sense when we sincerely judge someone says *"*now *the time is ...,"* or simply *"the time is ...,"* to be very specific to the point of second. Whenever it is said, the truth-value of it is hard to tell. Whenever the utterance is completed, it is almost a false utterance.

6.2 Moorean sentences

A Moorean sentence not only causes problems when it is uttered; it also generates puzzlement when put into an epistemic logic system that concerns knowledge of the sentence. It turns out that a Moorean sentence, which is not a contradiction, can't be consistently known. That is, $\sim K(p \,\&\!\sim Kp)$ is logically valid. Why after all we can't know a non-contradictory sentence? What prevents us from knowing it? Researchers have not yet tried to figure out why this is the case, but instead have directly utilized it in other philosophical programs. Hintikka takes it as the source of the absurdity of Moore's paradox [8], contending that it's a pragmatic norm not to utter something that couldn't be consistently known, and Fitch uses it to derive the knowability paradox, arguing that innocent verification principle leads to that all truth are known [6, 16]. I think the reason why the Moorean sentences can't be consistently known haven't been serious discussed is due to lack of machinery. Here I will use tMEL^{∞}, trying to figure out what's going on out there

First, it is possible to know Moorean types of sentences (in the following i, j are natural numbers) :

Lemma 6.1. *There is a* $\mathsf{tS4}^\infty$ *logical base* $\mathcal{A}$ *weak enough such that* $K^i(p\,\&{\sim}K^j p)$ *is* $\mathsf{tS4}^\infty(\mathcal{A})$ *satisfiable.*[16]

Proof. Consider the empty base $\mathcal{A}$. But actually any logical base with which the agent could not simplify a conjunction can do. Then we can create a set of U just like the one in the proof of Lemma 5.1, but adding $p\,\&{\sim}K^j p$ for some j to the set at some stage. Notice that since $\mathcal{A}$ is so weak that p won't be in U. Let M be a single world $\mathsf{tS4}^\infty(\mathcal{A})$-structure where the $\mathcal{A}$ awareness function at the world is the one created from U, and p is true at the world. $(M,w) \vDash p\,\&{\sim}K^j p$, and hence $(M,w) \vDash K^i(p\,\&{\sim}K^j p)$ for some i. □

But suppose the agent is smart enough to be able to simplify an conjunction. Then still the agent can know some Moorean sentences. Let's call a $\mathsf{tS4}^\infty$ logical base $\mathcal{A}$ such that for every $\mathcal{A}$ awareness function α, if $\alpha(\phi\,\&\psi){=}n$, then $\alpha(\phi){=}m$ for some m with $n{<}m$ a *smart enough* logical base.

Lemma 6.2. *For any smart enough* $\mathsf{tS4}^\infty$ *logical base* $\mathcal{A}$*, there are* i, j *such that* $K^i(p\,\&{\sim}K^j p)$ *is* $\mathsf{tS4}^\infty(\mathcal{A})$ *satisfiable.*

Proof. Again define U and the structure M just like the way they are defined in the above lemma. But now p will be in U at some stage, say m later than the stage, say n, to which the formula $p\,\&{\sim}K^j p$ is added. Now even so, $(M,w) \vDash p\,\&{\sim}K^j p$ for $j{\le}n$, hence $(M,w) \vDash K^i(p\,\&{\sim}K^j p)$, for $i{\ge}m$ be larger or equal to the stage number in which $p\,\&{\sim}K^j p$ is added to U. □

Now is a result of inconsistent knowledge.

Lemma 6.3. *For any smart enough* $\mathsf{tS4}^\infty$ *logical base* $\mathcal{A}$*,* $K^i(p\,\&{\sim}K^\infty p)$ *is not* $\mathsf{tS4}^\infty(\mathcal{A})$ *satisfiable.*

Proof. Let $M{=}\langle W, R, \mathfrak{A}, \mathcal{V}\rangle$ be the $\mathsf{tS4}^\infty(\mathcal{A})$-structure such that for some $w{\in}W$ $(M,w) \vDash K^i(p\,\&{\sim}K^\infty p)$. Then $\alpha_w(p\,\&{\sim}K^\infty p){\le}i$ for the $\mathcal{A}$ awareness function α_w. But then $\alpha_w(p){\downarrow}$, and $p\,\&{\sim}K^\infty p$ is not satisfiable at w. Neither is $K^i(p\,\&{\sim}K^\infty p)$. □

Here is a general result concerning $\mathsf{tS4}^\infty$.

Lemma 6.4. *For any* $\mathsf{tS4}^\infty$ *logical base* $\mathcal{A}$*, if for any* i*,* ${\sim}K^i\phi$ *is* $\mathsf{tS4}^\infty(\mathcal{A})$ *valid, then* ${\sim}K^\infty\phi$ *is* $\mathsf{tS4}^\infty(\mathcal{A})$ *valid.*

From these lemmas, we can derive that ${\sim}K^\infty(p\,\&{\sim}K^\infty p)$ is $\mathsf{tS4}^\infty(\mathcal{A})$ valid for smart enough logical bases $\mathcal{A}$.

In all the arguments given here, we don't really need the possible world machinery but mostly play with the awareness functions. So in a way, the statement is valid because of the relation of the labels. It is because the label of the first epistemic modality (∞ on the left) of the sentence is smaller than the label of the second modality (∞ on the right); or we can consider ${\sim}K^\infty(p\,\&{\sim}K^\infty p)$ is the limit of the sequences of formulas ${\sim}K^i(p\,\&{\sim}K^\infty p)$. In either way, it means at any time we could not correctly predict that something is true but we could not know it in the distant future. A meaningful statement.

Some more words on the issue. When we set a limitation on the domain of epistemic logic such that the basic events, events represented by propositional

[16] Just consider & as a derived connective in our system.

letters, are general events, whose truth-values would not change over time, it simplifies things a lot. But even so, when we take seriously that knowledge will evolve, the truth-values of knowledge sentences vary with time; so things couldn't be too simplified. Consider a higher-order knowledge sentence $K\phi$ in which ϕ contains other knowledge sentences. Then the variation of $K\phi$ in the truth-value as the time goes by depends not only on the agent's change of her/his state of mind, but also on the variations of the truth-values of ϕ. Now regarding the knowledge of Moorean sentences, which is certainly higher-order knowledge, in the course of the agent's reasoning, we have different cases to examine. Suppose p is false, $p \,\&{\sim}Kp$ can't be true and hence ${\sim}K(p \,\&{\sim}Kp)$ is true of the agent under discussion. Then suppose p is true, and in the beginning of the reasoning the agent doesn't know p and doesn't know the Moorean sentence $p \,\&{\sim}Kp$, we have the following three cases of what's happening in the course of the agent's reasoning:

Case 1. The agent doesn't learn the truth of p, and doesn't learn that $p \,\&{\sim}Kp$, which will be true in the whole course of reasoning.
So in the whole course of reasoning the agent doesn't know that $p \,\&{\sim}Kp$.

Case 2. Suppose the agent at some point learns that p is true, but not that $p \,\&{\sim}Kp$, even if it is true.
Then before the point, $p \,\&{\sim}Kp$ is true, but since the agent doesn't learn this, so ${\sim}K(p \,\&{\sim}Kp)$. After that point, since $p \,\&{\sim}Kp$ is false, so again ${\sim}K(p \,\&{\sim}Kp)$. Thus at the end, the agent doesn't know that $p \,\&{\sim}Kp$.

Case 3. Suppose the agent at some point learns that $p \,\&{\sim}Kp$ but doesn't learn that p at the point yet
Before this point, ${\sim}K(p \,\&{\sim}Kp)$, and just right after this point the agent knows that $p \,\&{\sim}Kp$. However, assume the agent is smart enough. Then p will be known at some other point later. Then before this latter point, still $K(p \,\&{\sim}Kp)$, but after that, since ${\sim}Kp$ is false, so at the end, the agent doesn't know that $p \,\&{\sim}Kp$.

Remember that in the argument of the invalidity of knowing a Moorean sentence with the possible world semantics, we argue that since if $(M, w) \vDash K(p \,\&{\sim}Kp)$ for some structure and some world, then $(M, w') \vDash p \,\&{\sim}Kp$ for all accessible worlds w' from w, so it is impossible $(M, w') \vDash {\sim}Kp$, given that p is true in all the accessible worlds. However, even given that p is true, there are still cases to consider about the conditions of the agent's reasoning.

In all these cases we discussed above, agents don't know that $p \,\&{\sim}Kp$ at the end, but they don't know it for different reasons. Furthermore, in Case 3, the agent indeed knows the Moorean sentence for some time. Also notice that a sentence of the form $p \,\&{\sim}Kq$, where q is a logical consequence of p, can't be consistently known either. The above analysis can be applied to this sentences as well. Consider Case 3. Then even though eventually the agent doesn't know that $p \,\&{\sim}Kq$, the agent can know this for quite some time in the course of the reasoning; the magnitude of the period of time depends on how long for the agent to reason in order to know q after the agent learns that p is the case. This is where we know Moorean sentences, which are not self-falsified.

The technical details in this paper is not that dramatic. But the goal of this paper is to bring out the idea that a logical analysis of knowledge should include the temporal aspects that reflects the amount of time that an agent needs in order to deduce to have new knowledge. For this idea, we introduce systems of epistemic logic of this kind, and show you with examples that this analysis can shed some light on some old riddles and puzzles in epistemic reasoning.

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